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REARRANGEMENT OF COEFFICIENTS OF FOURIER SERIES ON SU_2*

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We study non-commutative analogues of the following result: "Suppose $f \sim \sum a_n e^{int}$ is an integrable function on the circle which remains in L^1 no matter however we permute the coefficients a_n, then f belongs to L^2".

1. S. Helgason [3] proved the following theorem and asked whether a generalization to arbitrary compact groups holds.

THEOREM 1 (Helgason). *Let G be a compact abelian group with dual object $\hat{G}$, and let $\sum_{\lambda \in \hat{G}} a_\lambda \gamma_\lambda(x)$ be the Fourier series of an integrable function $f(x)$. Suppose that for any permutation σ of the elements of $\hat{G}$ the series $\sum a_{\sigma(\lambda)} \gamma_\lambda(x)$ still represents an L^1-function. Then f belongs to $L^2(G)$.*

Using a result of Coifman and Weiss it is easy to show that the theorem fails for arbitrary compact groups. However, after a more careful observation, the problem reveals several aspects. Moreover it turns out to be connected with the study of the L^p-divergence of Fourier series as considered in Coifman and Weiss [1] for SU_2 and in Giulini, Soardi and Travaglini [2] for compact connected semisimple Lie groups.

We shall study explicitly the case $G = SU_2$ and we shall consider L^p and L^q $(1 \leqslant p < q \leqslant \infty)$ in place of L^1 and L^2 respectively. Roughly speaking we show that for central functions on SU_2 the generalization of the theorem fails for $p < \frac{3}{2} \leqslant q$, it holds for $\frac{3}{2} < p < q \leqslant 2$, it fails for $\frac{3}{2} < p \leqslant 2 < q$ and it is meaningless for $p \geqslant 3$ (i.e. only the trigonometric polynomials remain in L^3 for any permutation of the coefficients).

Then we show that for arbitrary $L^p(SU_2)$-functions a generalization of the theorem fails for any $p < 2$ and $q \geqslant 2$ (in the non-central case the definition of "permutation" is somewhat more arbitrary, and one could consider other possibilities).

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Some of the above results extend easily to any compact simple simply connected Lie group, but in order to avoid introducing more technical notation we shall not state these generalizations.

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2. We recall (cf. Coifman and Weiss [1]) that the dual object Γ of SU_2 can be realized explicitly in the space of homogeneous polynomials of degree $2n$, where n runs through the non-negative half-integers. A maximal torus T of SU_2 is 1-dimensional, and to any equivalence class n of unitary irreducible representations we associate the character χ_n, whose restriction to T has the form $\chi_n(t) = \sin(2n+1)t/\sin t$. For any central integrable function f we have $\int_{SU_2} f = \pi^{-1} \int_{-\pi}^{\pi} f(t)\sin^2 t\,dt$. If U_n denotes a representation in the equivalence class n, we have the following Fourier expansion for central and arbitrary functions on SU_2 respectively:

$$f \sim \sum_{2n=0}^{\infty} a_n \chi_n \quad \left(a_n = \int_{SU_2} f(g)\overline{\chi_n(g)}\,dg\right),$$

$$f \sim \sum_{2n=0}^{\infty} (2n+1)\operatorname{Trace}(A_n U_n) \quad \left(A_n = \int_{SU_2} f(g)\,U_n(g^{-1})\,dg\right).$$

Definition. We say that a central function $f \sim \sum a_n \chi_n$ *remains in* $L^p(SU_2)$ *under any permutation* if for any permutation σ of the elements of Γ the series $\sum a_{\sigma(n)} \chi_n$ represents a function in L^p.

LEMMA. *For any* n, $\|\chi_n\|_3 \geqslant \text{const}\cdot(\log n)^{1/3}$.

Proof. Computation (or see [2] for a more general case).

We can state our first result.

THEOREM 2. (a) *There exists a central function not belonging to* $L^{3/2}(SU_2)$ *which remains in* L^p *under any permutation, for any* $p < \frac{3}{2}$.

(b) *Let* $\frac{3}{2} < p < q \leqslant 2$. *Suppose that a central function* f *remains in* L^p *under any permutation. Then* f *belongs to* L^q.

(c) *Let* $\frac{3}{2} < p \leqslant 2 < q$. *Then there exists a central function not belonging to* $L^q(SU_2)$ *which remains in* L^p *under any permutation.*

(d) *For* $p \geqslant 3$ *only the trigonometric polynomials remain in* L^p *under any permutation.*

Proof. (a) Let h be the function on SU_2 obtained by central extension of the even function $\frac{1}{2}\sin^{-2} t$ on T. Then, as pointed out in Coifman and Weiss [1, p. 127], Weyl's formulas and an orthogonality argument show that h has the Fourier expansion $h \sim \sum_{2n=0}^{\infty} \chi_n$ and a computation shows that h belongs to $L^p(SU_2)$ if and only if $p < \frac{3}{2}$.

(b) Let $f \sim \sum a_n \chi_n$ belong to $L^p(SU_2)$ for $\frac{3}{2} \leqslant p \leqslant 2$. Consider the function $g(t) \sim \sum a_n \sin(2n+1)t$ defined on the circle T. We show that g belongs to $L^1(T)$. Indeed, by Hölder's inequality and Weyl's formulas,

$$\int_{-\pi}^{\pi} |g(t)|\, dt = \int_{-\pi}^{\pi} |\sin t|^{(p-2)/p} \cdot \left|\sum a_n \sin(2n+1)t\right| \cdot |\sin t|^{(2-p)/p}\, dt$$

$$\leqslant \Big(\int_{-\pi}^{\pi} |\sin t|^{(p-2)/(p-1)}\, dt\Big)^{(p-1)/p} \cdot \Big(\int_{SU_2} |f(g)|^p\, dg\Big)$$

$$\leqslant \mathrm{const}_p \cdot \|f\|_{L^p(SU_2)}.$$

Now, suppose that $\sum a_{\sigma(n)} \chi_n$ represents a function in $L^p(SU_2)$ for any choice of σ. Then $\sum a_{\sigma(n)} \sin(2n+1)t$ represents a function in $L^1(T)$ for any choice of σ. Therefore, by Theorem 1, $g \sim \sum a_n \sin(2n+1)t$ represents a function in $L^2(T)$.

The equality

$$\|f\|_{L^2(SU_2)} = \pi^{-1/2} \|g\|_{L^2(T)}$$

ends this part of the proof.

(c) Suppose (c) is false. Then the space of those central functions which remain in $L^p(SU_2)$ under any permutation should be a subspace of L^q; but $q > 2$ and, by (b), the space of central functions which remain in L^p under any permutation is exactly the space of central L^2-functions.

(d) Let $f \sim \sum a_n \chi_n$ belong to $L^3(SU_2)$. There exist an infinite subsequence $\{a_{n_k}\}$ of non-zero coefficients, a sequence $\{m_k\}$ in Γ, a constant H, a sequence of trigonometric polynomials $V_k = \sum v_n^{(k)} \chi_n$ and a permutation σ such that for any k

(i) $\sigma(m_k) = n_k$;

(ii) $\log m_k \geqslant a_{n_k}^{-6}$;

(iii) $v_n^{(k)} = 2n+1$ for $n \leqslant m_k$;

(iv) $v_n^{(k)} = 0$ for $n \geqslant m_{k+1}$;

(v) $\|V_k\|_1 \leqslant H$;

(vi) $$\sum_{m_{k-1} \leqslant n \leqslant m_{k+1},\, n \neq m_k} \left| a_{\sigma(n)} \cdot \left(\frac{v_n^{(k)} - v_n^{(k-1)}}{2n+1} \right) \right| \leqslant (\log m_{k+1})^{-4}$$

(condition (vi) can be satisfied since first we choose m_1 satisfying (ii), then we fix V_1 and finally we use the fact that $a_n \to 0$ to choose small coefficients which satisfy (vi) for $k = 1$, then we choose m_2, V_2 and another part of the permutation in order to satisfy (vi) for $m = 2$, and we go on). Condition (vi) assures us that for k large $\|(V_k - V_{k-1}) * f\|_3 \geqslant \mathrm{const} \cdot \|a_{\sigma(m_k)} \chi_{m_k}\|_3$. Then (i)–(vi) and the Lemma give

$$\left\|\sum a_{\sigma(n)} \chi_n\right\|_3 \geqslant \mathrm{const} \cdot \|(V_k - V_{k-1}) * f\|_3$$

$$\geqslant \mathrm{const} \cdot \|a_{\sigma(m_k)} \chi_{m_k}\|_3 \geqslant \mathrm{const} \cdot a_{n_k} \cdot (\log m_k)^{1/3}$$

$$\geqslant \mathrm{const} \cdot a_{n_k}^{-1} \to \infty \qquad (\text{as } k \to \infty).$$

Hence $\left\|\sum a_{\sigma(n)} \chi_n\right\|_3$ does not belong to $L^3(SU_2)$.

Remarks. Compare the techniques used in (a) and (d) with the proof of the divergence of Fourier series as in [1, p. 139] and [2]. As a consequence of the proof of (b) we observe that $a_n \to 0$ for any $f \sim \sum a_n \chi_n$ in $L^p(SU_2)$, $p > \frac{3}{2}$. The converse is true too; moreover a central function $f \in L^{3/2}(SU_2)$ may have unbounded coefficients. Indeed the Lemma implies, for any n, the existence of a function $f_n \sim \sum a_k^{(n)} \chi_k$ such that $\|f_n\|_{3/2} = 1$ and

$$\mathrm{const} \cdot (\log n)^{1/3} \leqslant \|\chi_n\|_3 = \int_{SU_2} f_n \bar{\chi}_n = a_n^{(n)}.$$

Then, adding in a suitable way some of these functions it is easy to construct a function $f \sim \sum a_n \chi_n$ such that $\|f\|_{3/2} = 1$ and the sequence $\{a_n\}$ is unbounded.

The reader familiar with Fourier analysis on compact Lie groups should expect only negative results when we drop the assumption f central from Theorem 2. Indeed we have a result in this sense, but first we must define properly the permutation in the non-central case, since we cannot exchange matrices of different dimensions. Let $f \sim \sum (2n+1) \mathrm{Trace}(A_n U_n)$; we can write in a unique way any matrix A_n as $A_n = a_n A_n^0$, where a_n is a complex number, the Hilbert–Schmidt norm of A_n^0 is $(2n+1)^{-1}$ and the trace of A_n^0 is non-negative. Then we write

$$f \sim \sum a_n \mathrm{Trace}(A_n^0 U_n)$$

and we say that f remains in $L^p(SU_2)$ under any permutation when for any permutation σ of the elements of Γ the series $\sum a_{\sigma(n)} \mathrm{Trace}(A_n^0 U_n)$ represents a function in L^p.

THEOREM 3. *There exists a function not belonging to $L^2(SU_2)$ which remains in L^p under any permutation, for any $p < 2$.*

Proof. We write $g \in SU_2$ as $g = \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix}$. It is known that for any equivalence class $n \in \Gamma$ there exists a representation U_n which admits α^{2n} as its (1, 1)-matrix coefficient. Then we consider

$$f(g) = (1-\alpha)^{-1} \sim \sum_{2n=0}^{\infty} \alpha^{2n} = \sum \mathrm{Trace}(A_n^0 U_n)$$

where the (1, 1)-coefficient of A_n^0 is $(2n+1)^{-1}$ and the others are zero. It is easy to check that f belongs to $L^p(SU_2)$ if and only if $p < 2$.

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