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RADICAL IRREGULARITY OF SOME POLYNOMIAL RINGS

BY

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1. Introduction. In [1] Gorman gave an example of a non-trivial differential ring of polynomials over a Ritt algebra, with an infinite set of variables, which is a radically regular ring.

In the present paper we prove at first that there is no non-trivial differential ring of polynomials over a differential ring being an integral domain with characteristics $p > 0$, which is a radically regular ring. Then we show that there is also no non-trivial differential ring of polynomials in finite number of variables over a Ritt algebra being an integral domain with finite Krull dimension, which is a radically regular ring.

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2. Preliminary notions. A *differential ring* is a pair (R, d), where R is a commutative ring with unit and $d: R \to R$ is a mapping, called *derivation*, satisfying the conditions

$$d(r+s) = d(r)+d(s) \quad \text{and} \quad d(rs) = rd(s)+sd(r)$$

for arbitrary $r, s \in R$.

Let (R, d) be a differential ring. An ideal U in R is called *differential* if $d(U) \subseteq U$. For an arbitrary $T \subset R$, we denote by (T), $[T]$ and $\{T\}$ the smallest ideal, the smallest differential ideal and the smallest radical differential ideal containing the set T, respectively. For an arbitrary Noetherian differential ring, any radical differential ideal is an intersection of a finite number of prime differential ideals (see [3]). (R, d) is called a *Ritt algebra* if R contains the field Q of rational numbers. In Ritt algebras any maximal differential ideal is prime (see [2]). (R, d) is called a *radically regular ring* if, for any $r \in R$, $\{r\} = R$ implies $(r) = R$ (see [1] and [2]). In particular (see [2], Lemma 4), a Ritt algebra is a radically regular ring if and only if, for any $r \in R$, $[r] = R$ implies $(r) = R$. Let $S = R[x_t\colon t \in T]$ be a polynomial ring over (R, d). The derivation d can be extended to S by setting, for $d(x_t)$, some polynomial $f_t \in S$ $(t \in T)$. We say that (S, d)

is a *non-trivial differential ring* if $d(x_{t_0}) \neq 0$ for some $t_0 \in T$. For any polynomial $g \in Q[x]$, the n-th derivative of g will be denoted by $g^{(n)}$.

3. Polynomial rings over rings with characteristic $p > 0$.

PROPOSITION 1. *Let (R, d) be a differential ring being an integral domain with characteristic $p > 0$ and let $S = R[x_t: t \in T]$ be a non-trivial differential ring of polynomials over (R, d). Then (S, d) is not a radically regular ring.*

Proof. There exists a variable x_{t_0} such that $d(x_{t_0}) \neq 0$. Put $x_{t_0} = x$ and $d(x) = f$ and consider the element xf^p+1. Obviously, the ideal (xf^p+1) is distinct from R. Since the element $f^{p+1} = d(xf^p+1)$ belongs to the radical ideal $\{xf^p+1\}$, so does f; hence $\{xf^p+1\} = R$.

4. Polynomial rings over a Ritt algebra. Let (R, d) be a Ritt algebra being an integral domain with finite Krull dimension (as a commutative ring) and let $S = R[x_1, \ldots, x_m]$ $(m \geqslant 1)$ be a non-trivial differential ring of polynomials in finite number of variables over (R, d). Assume that $x_1 = x$ and $d(x) = f \neq 0$. We define the sequence $u_0, u_1, \ldots$ of elements from S as follows:

$$u_0 = -1,$$
$$u_n = fd(u_{n-1}) - (2n-1)u_{n-1}d(f) \quad \text{for } n \geqslant 1.$$

LEMMA 1. *Let g be a polynomial from $Q[x]$, U a differential ideal in (S, d), and n a natural number. If $u_n + gf^{2n+1}$ belongs to U, then so does $u_{n+1} + g^{(1)}f^{2n+3}$.*

Proof. Since

$$fd(u_n + gf^{2n+1}) = fd(u_n) + (2n+1)gf^{2n+1}d(f) + g^{(1)}f^{2n+3}$$

belongs to U, so does the element

$$u_{n+1} + g^{(1)}f^{2n+3} = fd(u_n) - (2n+1)u_n d(f) + g^{(1)}f^{2n+3}.$$

LEMMA 2. *For any natural number n and any polynomial $g \in Q[x]$ the element $u_n + g^{(n)}f^{2n+1}$ belongs to the ideal $[gf-1]$.*

Proof. For $n = 0$, $u_0 + g^{(0)}f^{2\cdot 0+1} = -1+gf$ is clearly in $[gf-1]$. Assume that $u_n + g^{(n)}f^{2n+1}$ belongs to $[gf-1]$; then, by Lemma 1, $u_{n+1} + g^{(n+1)}f^{2n+3}$ is in $[gf-1]$.

LEMMA 3. *Let w be a polynomial in $Q[x]$, n any fixed natural number, and T an infinite subset of Q. Moreover, for any $t \in T$ let $u_n + (w+t)f^{2n+1}$ belong to one of non-zero prime differential ideals P_t in S, and let $f \notin P_t$. Then there exists a prime differential ideal P which contains $u_{n+1} + w^{(1)}f^{2n+3}$ and is essentially contained in some ideal P_t.*

Proof. Since $u_n + (w+t)f^{2n+1}$ is in P_t for any $t \in T$, Lemma 1 implies that the polynomial

$$u_{n+1} + w^{(1)}f^{2n+3} = u_{n+1} + (w+t)^{(1)}f^{2n+3}$$

belongs to all P_t, and so it is in a differential ideal

$$I = \bigcap_{t\in T} P_t.$$

I is a radical differential ideal in a Noetherian ring S and, consequently, $I = I_1 \cap \ldots \cap I_k$, where I_j $(1 \leqslant j \leqslant k)$ are prime differential ideals. Since $P_t \supseteq I$ for all $t \in T$, each P_t contains some ideal I_{i_t} $(1 \leqslant i_t \leqslant k)$. The set T is infinite, and so there exist two different elements $s, t \in T$ such that $I_{i_s} = I_{i_t}$. Put $P = I_{i_s} = I_{i_t}$; then, clearly $P_s \supseteq P$ and $P_t \supseteq P$. If $P = P_s$ and $P = P_t$, then $u_n + (w+s)f^{2n+1}$ and $u_n + (w+t)f^{2n+1}$ belong to P_t. Thus $(s-t)f^{2n+1}$ is in P_t, and so f belongs to P_t, contrary to the assumption. Hence P is essentially contained in one of the ideals P_t and P_s and, obviously, $u_{n+1} + w^{(1)}f^{2n+3}$ belongs to P.

THEOREM 1. *If (R, d) is a Ritt algebra being an integral domain with finite Krull dimension and $S = R[x_1, \ldots, x_m]$ $(m \geqslant 1)$ is a non-trivial differential ring (S, d) of polynomials, then the ring (S, d) is not radically regular.*

Proof. Assume that (S, d) is radically regular.

(A) First we show that, for any polynomial $g \in Q[x]$ of degree greater than zero and for any natural number n, elements $u_n + g^{(n)}f^{2n+1}$ are non-zero polynomials. Assume that $u_n + g^{(n)}f^{2n+1} = 0$ for some n and put $h = g + x^n$. Since (S, d) is a radically regular ring and since the polynomial $hf-1$ is not invertible in S, there exists a prime differential ideal P such that $[hf-1] \subseteq P$. This fact and Lemma 2 imply that the element

$$u_n + h^{(n)}f^{2n+1} = u_n + g^{(n)}f^{2n+1} + (n!)f^{2n+1} = (n!)f^{2n+1}$$

belongs to P and, consequently, $f \in P$. But $hf-1 \in P$ implies $P = S$, which is impossible.

(B) Since the Krull dimension of R is finite, so is the Krull dimension of S. Let $\dim S = u$ and let v be a fixed natural number such that $v \geqslant u$. Clearly, $v \geqslant 1$.

We will write any polynomial $g = a_r x^r + \ldots + a_0$, where $a_i \in Q$, $a_r \neq 0$, in the form $g = g(a_r, \ldots, a_0)$ and its k-th derivative, for $k < r$, as $g^{(k)} = g_k(a_r, \ldots, a_k)$.

Now consider all polynomials $g \in Q[x]$ of degree v. Since for any $a_v, \ldots, a_0$, where $a_v \neq 0$, the polynomial $g(a_v, \ldots, a_0)f-1$ is not invertible in S, by the assumption that S is radically regular there exist prime differential ideals $P(a_v, \ldots, a_0)$ such that

$$g(a_v, \ldots, a_0)f-1 \in P(a_v, \ldots, a_0).$$

All these prime ideals do not contain f and, by part (A) of this proof, they are non-zero ideals. Having fixed elements $a_v, \ldots, a_1$ and taking $T = Q$ we see that the assumptions of Lemma 3 are satisfied. Thus there exists a prime differential ideal $P(a_v, \ldots, a_1)$ containing the element $u_1 + g_1(a_v, \ldots, a_1)f^3$ and essentially contained in some prime ideal $P\big(a_v, \ldots, a_1, b_0(a_v, \ldots, a_1)\big)$. Running over sequences $(a_v, \ldots, a_1)$, where $a_v \neq 0$, we get ideals $P(a_v, \ldots, a_1)$ for which all assumptions of Lemma 3 are satisfied. So we get new prime ideals $P(a_v, \ldots, a_2)$ and the inclusion

$$P(a_v, \ldots, a_2) \subset P\big(a_v, \ldots, a_2, b_1(a_v, \ldots, a_2)\big).$$

Thus

$$\begin{aligned} P(a_v, \ldots, a_2) &\subset P\big(a_v, \ldots, a_2, b_1(a_v, \ldots, a_2)\big) \\ &\subset P\Big(a_v, \ldots, a_2, b_1(a_v, \ldots, a_2), b_0\big(a_v, \ldots, a_2, b_1(a_v, \ldots, a_2)\big)\Big). \end{aligned}$$

Continuing this process we get the following sequence of $v+1$ prime ideals:

$$0 \subset P(b_v^0) \subset P(b_v^0, b_{v-1}^0) \subset \ldots \subset P(b_v^0, \ldots, b_1^0, b_0^0).$$

This contradicts $\dim S = u < v+1$.

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