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ON INDEPENDENCE OF AXIOMS OF A CERTAIN CLASS OF TERNARY RINGS

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In the present note we are concerned with a certain class of ternary rings, which we shall call ternary Lie-Jacobson rings. In particular, we shall prove that axioms of this class are independent.

Definition 1. A non-empty set R with a binary operation $+$ (addition) and a ternary operation f (multiplication) is called a *ternary ring* if (see, e.g., [5])

1^0 R is an abelian group with respect to addition,

2^0 multiplication is distributive with respect to addition, i.e.,

$$f(x_1+x_2, y, z) = f(x_1, y, z)+f(x_2, y, z), \tag{1}$$

$$f(x, y_1+y_2, z) = f(x, y_1, z)+f(x, y_2, z), \tag{2}$$

$$f(x, y, z_1+z_2) = f(x, y, z_1)+f(x, y, z_2). \tag{3}$$

Definition 2. A ternary ring R is called *associative* if, for any $x, y, z, u, t \in R$,

$$f(f(x, y, z), u, t) = f(x, f(y, z, u), t) = f(x, y, f(z, u, t)).$$

Definition 3. A ternary ring R is called a *ternary Lie-Jacobson ring* if there are satisfied the following axioms:

(i) $f(x, x, y) = 0$,

(ii) $f(x, y, z)+f(y, z, x)+f(z, x, y) = 0$,

(iii) $f(f(x, y, z), u, t)+f(f(y, x, u), z, t)+f(y, x, f(z, u, t))+$
$+f(z, u, f(x, y, t)) = 0$.

THEOREM 1. *If R is a ternary associative ring with addition $+$ and multiplication g, then the set R with the same addition $+$ and the multiplication f defined by the formula*

$$f(x, y, z) = g(x, y, z)-g(y, x, z)-g(z, x, y)+g(z, y, x)$$

is a Lie-Jacobson ring.

Proof consisting in easy but painstaking checking of conditions (1)-(3) and (i)-(iii) we leave to the reader.

A Lie-Jacobson ring obtained from a ternary associative ring R in the way described in theorem 1 will be denoted in the sequel by $R^{(-)}$.

If ring R in theorem 1 has been obtained from a binary ring by the definition of multiplication

$$g(x, y, z) = xyz,$$

then the multiplication in $R^{(-)}$ can be written as

$$f(x, y, z) = \big[[x, y], z\big], \tag{4}$$

where $[x, y] = xy - yx$ is a commutator of elements x, y. Formula (4) is the definition of a ternary multiplication in a binary Lie ring obtained in the well-know way from an associative ring.

Rings with multiplication (4) have been investigated by Jacobson [3] (see also [7]) in connexion with some problems of the theory of meson fields (see [2] and [4]). He called them *Lie Triple Systems*.

It seems worth to notice that between differentiation in a ternary associative ring R and differentiation in $R^{(-)}$ there is a connexion analogous to that between differentiation in a binary associative ring S and the Lie ring $S^{(-)}$ related to S (see [6] and [8]).

THEOREM 2. *In ternary ring axioms* (i), (ii) *and* (iii) *are independent.*

Proof. First we shall show that axiom (i) is independent of (ii) and (iii). For that purpose take free abelian semigroup with the generators a and b, and make of it a new semigroup S by adjoining zero; juxtaposition zero to any word yields zero and any word of length $\geqslant 4$ is equal to zero.

Hence

$$S = \{0, a, b, aa, ab, bb, aaa, aab, abb, bbb\}.$$

Now take a 9-dimensional semigroup algebra $\mathfrak{A}$ over Galois field GF (3), a base of which are all elements of semigroup S distinct from zero. Consider $\mathfrak{A}$ as a binary ring and define in it a ternary multiplication by

$$f(x, y, z) = xyz \qquad (x, y, z \in \mathfrak{A}).$$

In this way $\mathfrak{A}$ with the new multiplication becomes a ternary ring which obviously satisfies (ii) and also satisfies (iii), because product of any five elements of the algebra equals zero. But it does not satisfy (i): $f(a, a, b) \neq 0$.

To show that (ii) is independent of (i) and (iii) take a 7-dimensional Grassman algebra $\mathfrak{G}$ over Galois field $GF(2)$, with a base consisting of words

$$e_1, e_2, e_3, e_1 e_2, e_1 e_3, e_2 e_3, e_1 e_2 e_3.$$

Considering $\mathfrak{G}$ as a binary ring and introducing new multiplication

$$f(x, y, z) = xyz \quad (x, y, z \in \mathfrak{G}),$$

we get a ternary ring in which axiom (i) is satisfied. This follows from the fact that for each $x \in \mathfrak{G}$, in view of the definition of a Grassman algebra (cf. [1]) and properties of $GF(2)$, there is

$$x^2 = (a_1 e_1 + a_2 e_2 + a_3 e_3 + a_{12} e_1 e_2 + a_{13} e_1 e_3 + a_{23} e_2 e_3 + a_{123} e_1 e_2 e_3)^2 = 0,$$

where all a's belong to $GF(2)$.

Axiom (iii) is also satisfied, because product of any five elements of $\mathfrak{G}$ is equal to zero. However, $\mathfrak{G}$ does not satisfy (ii):

$$f(e_1, e_2, e_3) + f(e_2, e_3, e_1) + f(e_3, e_1, e_2) = f(e_1, e_2, e_3) = e_1 e_2 e_3 \neq 0.$$

Finally, it remains to show that (iii) is independent of (i) and (ii). To that end take the Cayley-Dicson algebra $\mathfrak{D}$ over $GF(3)$ considering it as a binary ring and introducing a ternary multiplication by $f(x, y, z) = (xy)z - x(yz)$ (f is an associator).

Since the Cayley-Dicson algebra is alternative, axiom (i) is satisfied. So is (ii), because

$$f(x, y, z) + f(y, z, x) + f(z, x, y) = 3f(x, y, z) = 0,$$

where the first equality follows from the well-known property of an associator in an alternative algebra.

However, axiom (iii) is not satisfied, for if

$$1, i, j, k, e, ie, je, ke$$

are elements of the canonical base of an 8-dimensional linear space which is the support of $\mathfrak{D}$, then multiplication of these elements in $\mathfrak{D}$ is described by the following table:

	1	i	j	k	e	ie	je	ke
1	1	i	j	k	e	ie	je	ke
i	i	2	k	$2j$	ie	$2e$	$2ke$	je
j	j	$2k$	2	i	je	ke	$2e$	$2ie$
k	k	j	$2i$	2	ke	$2je$	ie	$2e$
e	e	$2ie$	$2je$	$2ke$	2	i	j	k
ie	ie	e	$2ke$	je	$2i$	2	$2k$	j
je	je	ke	e	$2ie$	$2j$	k	2	$2i$
ke	ke	$2je$	ie	e	$2k$	$2j$	i	2

Hence

$$f(f(i,j,k)e, ie)+f(f(j,i,e)k, ie)+f(j,i,f(k,e,ie))+$$
$$+f(k,e,f(i,j,ie)) = f(0,e,ie)+f(ke,k,ie)+f(j,i,j)+f(k,e,je) = i.$$

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