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REMARKS ON A GENERALIZATION OF COMPLETENESS IN THE SENSE OF ČECH

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The aim of this paper is to discuss a generalization of completeness in the sense of Čech introduced recently by Timokhovich [9]. A Hausdorff space X will be said to be *complete in the sense of the Fomin H-closed extension* σX, shortly *F-complete*, if the remainder $\sigma X \setminus X$ is an F_σ-set in σX (see [9], Theorem 1). The Baire Theorem for F-complete spaces is discussed. Some results from [9] concerning subspaces, continuous images and preimages and products of F-complete spaces are strengthened.

All spaces are assumed to be Hausdorff and all maps continuous.

1. The Baire Theorem. The *Fomin H-closed extension* σX (see Fomin [4]) of a space X is the set $\sigma X = X \cup R_X$, where $R_X = \{\xi : \xi$ is an ultrafilter (1) without adherence points$\}$ with the topology generated by the sets

$$\sigma(U) = U \cup \{\xi \in R_X : U \in \xi\},$$

where U is open in X; this family forms a base in σX.

LEMMA 1. *If A, $A \subset \sigma X$, is closed and $A \cap X = \emptyset$, then A is compact.*

Proof. Let $\mathscr{P}$ be a cover of A which consists of basic sets. Since $\sigma X \setminus A$ is open, the family

$$\mathscr{V} = \mathscr{P} \cup \{\sigma(U) : \sigma(U) \cap A = \emptyset\}$$

forms an open cover of σX. The space σX is H-closed, hence there exists a finite subfamily $\{\sigma(U_1), \ldots, \sigma(U_n)\} \subset \mathscr{V}$ such that

$$\mathrm{cl}_{\sigma X}\sigma(U_1) \cup \ldots \cup \mathrm{cl}_{\sigma X}\sigma(U_n) = \sigma X.$$

It is known (see [1]) that $\mathrm{cl}_{\sigma X}\sigma(U) = \mathrm{cl}_X U \cup \sigma(U)$ for each U open in X. Hence $A \subset \sigma(U_1) \cup \ldots \cup \sigma(U_n)$, A being contained in the remainder. Clearly, $\sigma(U_i)$ belongs to $\mathscr{P}$ whenever $\sigma(U_i) \cap A \neq \emptyset$, $i = 1, 2, \ldots, n$. Thus A has a finite subcover which consists of elements of $\mathscr{P}$, which completes the proof.

(1) An *ultrafilter* means here an ultrafilter consisting of open sets.

Mioduszewski [7] proved the following version of the Baire Theorem:

An H-closed space X which is a countable union of compact spaces $A_1, A_2, \ldots$ does not contain any subset Y such that $Y \cap A_1$, $Y \cap A_2, \ldots$ are all compact and nowhere dense in Y.

From this theorem we infer

THEOREM 1. *An F-complete space which is a countable union of compact spaces $A_1, A_2, \ldots$ does not contain any subset Y such that $Y \cap A_1$, $Y \cap A_2, \ldots$ are all compact and nowhere dense in Y.*

Proof. It suffices to note that if an F-complete space X is a countable union of compact spaces, then, by Lemma 1, σX is a countable union of compact spaces.

Note. The Baire Theorem in the usual form asserts that no countable union of nowhere dense subsets has a non-empty interior. Such an assertion is not true for H-closed spaces (and all the more for F-complete spaces) and this was shown by Herrlich [6] (for another example, see also Mioduszewski [7]); even in the case of compact nowhere dense subsets it may happen that a countable union of such spaces has a non-empty interior (the same example).

But in the case of regular F-complete spaces the Baire Theorem holds in the usual form. A simple proof can be obtained by a modification of the proof in the case of spaces complete in the sense of Čech (see Engelking [3], p. 145).

2. Subspaces, images and preimages of F-complete spaces. A map $f\colon X \to Y$ is said to be *τ-proper* (see Mioduszewski and this author [2]) iff there exists a map $\tau f\colon \tau X \to \tau Y$ (τZ denotes the Katětov H-closed extension of Z) completing the diagram

$$\begin{tikzcd} X \arrow[d, "f"'] \arrow[r, phantom, "\subset"] & \tau X \arrow[d, "\tau f"] \\ Y \arrow[r, phantom, "\subset"] & \tau Y \end{tikzcd}$$

If, in addition, f maps the remainder into the remainder, i.e., if $\tau f(\tau X \setminus X) \subset \tau Y \setminus Y$, then f is said to be *τ-perfect.*

For the characterization of τ-proper maps in terms of covers, see Harris [5] (τ-proper maps are called there *p-maps*).

In [1] it was shown that in the definition of τ-perfectness the Katětov extension can be replaced by the Fomin extension.

THEOREM 2. *If a map $f\colon X \xrightarrow{\text{onto}} Y$ is τ-perfect, then Y is F-complete iff X is F-complete.*

The proof, by Lemma 1, is obvious.

An analogous fact under the assumption that f is perfect and spaces are regular was shown in [9]. But perfect maps need not be τ-perfect and, conversely, τ-perfect maps need not be perfect (see [2], p. 46). This means, e.g., that preimages of H-closed spaces under perfect maps are not necessarily H-closed. Later, we shall show that they are not necessarily F-complete.

Timokhovich proved in [9] that a closed subspace of a regular F-complete space is F-complete. If we do not assume regularity, then we have the following

THEOREM 3. *If X is F-complete, $A \subset X$ is closed and the inclusion is τ-proper, then A is F-complete.*

Proof. It is known from [2] that a τ-proper map $f\colon X \to Y$ is τ-perfect iff $f(F)$ is closed for each regularly closed $F \subset X$ and, for each $y \in Y$ and each ultrafilter ξ in X without adherence points, there exists a $U \in \xi$ such that $f^{-1}(y) \cap \mathrm{cl}\, U = \emptyset$. Hence the inclusion $A \subset X$ is τ-perfect. Since $\tau X \setminus X$ is an F_σ-set, so is $\sigma A \setminus A$. This completes the proof.

If A is a regularly closed set, then the inclusion $A \subset X$ is τ-proper (cf. [1]). Hence, by Theorem 3, we infer the following

COROLLARY. *A regularly closed subset of an F-complete space is F--complete.*

We shall show that Theorem 3 is not true without the assumption that the inclusion is τ-proper.

Example 1. Let $\mathcal{T}$ be the natural topology on the segment $I = [0, 1]$. Let $\mathcal{T}'$ be the topology on I generated by the family $\mathcal{T} \cup \{D\}$, where D denotes the set of all irrationals in I. It is easy to see that $\mathcal{T}'$ is an H-closed topology on I. Clearly, the set of all rationals is closed in $\mathcal{T}'$ and, by Theorem 1, it is not F-complete.

In the non-regular case the preimage of an H-closed space under perfect map is not necessarily F-complete.

Example 2. The previous example shows that there exists an H--closed space X which admits a closed, countable and dense in itself subset $A \subset X$ (hence X is not regular). Let Y be a disjoint union of A and X, i.e.,

$$Y = \{0\} \times A \cup \{1\} \times X.$$

Consider a map

$$f = p_X | Y\colon\ Y \xrightarrow{\text{onto}} X,$$

where $p_X\colon \{0, 1\} \times X \to X$ is a projection. It is easy to check that f is perfect. Suppose that Y is F-complete. Then, by the corollary, A is F-complete. Now, a contradiction with Theorem 1 is easy to observe.

3. Cartesian product of F-complete spaces. In [9] it was proved that finite products of F-complete spaces are F-complete. We shall show that

THEOREM 4. *The Cartesian product of a countable family of F-complete spaces is F-complete.*

Proof. $X_1, X_2, \ldots$ be F-complete. One can prove (see [9], Theorem 1) that a space X is F-complete iff there exists a countable family $\mathscr{P}_1, \mathscr{P}_2, \ldots$ of open covers such that every ultrafilter containing at least one member of each cover $\mathscr{P}_i$ for $i = 1, 2, \ldots$ has a limit point (in [9] this property is stated as a definition of F-completeness). Hence, for each X_n, there exists a countable family of covers $\mathscr{P}_1^n, \mathscr{P}_2^n, \ldots$ such that each ultrafilter in X_n containing at least one element of $\mathscr{P}_k^n$ for $k = 1, 2, \ldots$ has a limit point. Let X be the Cartesian product of spaces $X_1, X_2, \ldots$ and let $p_n\colon X \to X_n$ be the projections. Consider the family $\mathscr{Q}_k^n = \{p_n^{-1}(U)\colon U \in \mathscr{P}_k^n\}$. Clearly, $\{\mathscr{Q}_k^n\colon n, k = 1, 2, \ldots\}$ is a countable family of open covers of X. Let ξ be an ultrafilter in X which contains members of $\mathscr{Q}_k^n$ for each n, $k = 1, 2, \ldots$ The family $\xi_n = \{p_n(V)\colon V \in \xi\}$ is an ultrafilter in X_n for every integer n. It is easy to see that ξ_n contains at least one member of each $\mathscr{P}_k^n$, where $k = 1, 2, \ldots$ Hence there exists a point $x_0 \in X$ such that $p_n(x_0)$ is a limit point of ξ_n for every n. It is easy to check that x_0 is a limit point of ξ, which completes the proof.

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