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ON INVARIANT FUNCTIONS AND ERGODIC MEASURES OF MARKOV OPERATORS ON $C(X)$

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Let $C(X)$ be the Banach space of all real-valued continuous functions on a compact Hausdorff space X. A linear operator T on $C(X)$ is called *Markov* if T is positive ($f \geqslant 0 \Rightarrow Tf \geqslant 0$) and $T1 = 1$. We denote by $C_T(X)$ the subspace of all T-invariant functions in $C(X)$, and by $P_T(X)$ the subspace of all T^*-invariant probability (Radon) measures on X. By the Markov–Kakutani fixed point theorem, $P_T(X)$ is a non-empty convex and w^* compact subset of $C^*(X)$. A measure $\mu \in P_T(X)$ is called *ergodic* if μ is an extreme point of $P_T(X)$. The set of all ergodic measures is denoted by $\operatorname{ex} P_T(X)$.

A Markov operator T is called *strong mean ergodic* (s.m.e.) if for every $f \in C(X)$ the Cesàro means

$$A_n f = n^{-1}(f + Tf + \ldots + T^{n-1} f)$$

converge in $C(X)$. Then the strong operator limit $P = \lim A_n$ satisfies $TP = PT = P$. In [7] Sine proved that T is s.m.e. iff $C_T(X)$ separates $P_T(X)$.

In the sequel we shall use the concept of Bauer simplex. Let K be a compact convex subset of a locally convex topological vector space E. The set

$$\tilde{K} = \{\alpha x: \alpha \geqslant 0, x \in K\}$$

is the cone generated by K, so it induces a translation invariant partial ordering on E:

$$x \geqslant y \quad \text{iff} \quad x - y \in \tilde{K}.$$

K is called a *simplex* if the space $\tilde{K} - \tilde{K}$ is a lattice in the ordering induced by $\tilde{K}$.

By the Choquet–Meyer Theorem (see [3], p. 66) and by Theorem II.3.6 of [1] we see that the above definition of a simplex is equal to the definition of a Choquet simplex from [1]. If T is a Markov operator, then $P_T(X)$ is a simplex (see, e.g., [5]). A set K is called a *Bauer simplex* if K is a simplex and the set of all extreme points of K is closed (see [1], p. 103).

Let $A(K)$ denote the space of all continuous affine functions on K with the supremum norm. By the Bauer Theorem, K is a Bauer simplex iff $A(K)$ is a lattice in the natural ordering of functions (see [1], Theorem II.4.1).

1. Invariant functions. Let T be a Markov operator on $C(X)$. Recall that T is called *uniquely ergodic* iff the set $P_T(X)$ consists of exactly one measure. It is known that T is then s.m.e. If $P_T(X) = \{\mu\}$, then the associated projection is of the form

$$Pf = (\mu, f)\,1,$$

so $C_T(X)$ is a sublattice of $C(X)$. In particular, it follows from Theorem 1 in [4] that the set of Markov operators T for which $C_T(X)$ forms a sublattice of $C(X)$ is norm residual in the space of all Markov operators on $C(X)$. We note that, in general, $C_T(X)$ need not form a lattice in $C(X)$.

Example. First we consider the Markov operator S on

$$Y = \{0\} \cup \{\pm 3^{-n},\ n = 0, 1, 2, \ldots\}$$

defined by the formula

$$S^* \delta_{\pm 3^{-n}} = 2^{-1} \delta_{\mp 3^{-n}} + 2^{-1} \delta_{\pm 3^{-(n-1)}}$$

for $n = 1, 2, \ldots$,

$$S^* \delta_{\pm 1} = \delta_{\pm 1}, \qquad S^* \delta_0 = \delta_0.$$

It is not difficult to show that $C_S(Y)$ is equal to the space of all linear functions on $[-1, 1]$ restricted to Y. Therefore, $C_S(Y)$ is a lattice but not a sublattice of $C(Y)$.

Now we modify the example and obtain $C_T(X)$ not even being a lattice in $C(X)$.

Let $X = Y_1 \cup Y_2$, where $Y_1 = Y \times \{0\}$, $Y_2 = \{0\} \times Y$. On $C(X)$ we define a Markov operator T as follows:

$$T^* \delta_{(\pm 3^{-n}, 0)} = 2^{-1} \delta_{(\mp 3^{-n}, 0)} + 2^{-1} \delta_{(\pm 3^{-(n-1)}, 0)},$$
$$T^* \delta_{(0, \pm 3^{-n})} = 2^{-1} \delta_{(0, \mp 3^{-n})} + 2^{-1} \delta_{(0, \pm 3^{-(n-1)})}$$

for $n = 1, 2, \ldots$,

$$T^* \delta_x = \delta_x \quad \text{for } x = (\pm 1, 0),\ (0, \pm 1),\ (0, 0).$$

Then f belongs to $C_T(X)$ iff $f|_{Y_i} = f_i$ for some f_i in $C_S(Y_i)$, $i = 1, 2$, and $f_1(0, 0) = f_2(0, 0)$. It suffices to note that $C_T(X)$ is not a lattice in $C(X)$.

In general, it seems to be difficult to check that $C_T(X)$ is a lattice in $C(X)$, since (in general) it cannot be known what the modulus is. For example, the condition that for every $f \in C_T(X)$ there exists $\lim A_n |f|$ in $C(X)$ is sufficient for $C_T(X)$ to be a lattice. Here $\operatorname{mod} f = \lim A_n |f|$. From the point of view of the theory of Markov operators it appears to be easier to investigate when $C_T(X)$ is a sublattice of $C(X)$.

Let M be the closure of the union of the supports of all invariant probabilities. M will be called the *center* of T. For example, if $M = X$ or T is s.m.e. and the associated projection P is strictly positive (i.e., $f \geqslant 0$, $f \neq 0 \Rightarrow Pf \neq 0$), then $C_T(X)$ is a sublattice of $C(X)$. This is a consequence of Proposition 11.5, Chapter III in [6], and Lemma 1.5 in [7]. The Markov operator $Tf(x) = f(x^2)$ on $C[0, 1]$ shows that $C_T(X)$ is a sublattice of $C(X)$ but $M \neq X$. To see that the second implication cannot be reversed, we consider any uniquely ergodic Markov operator T with the invariant probability μ such that the $\operatorname{supp}\mu$ is a proper subset of X. Clearly, T is s.m.e. and $C_T(X)$ is a sublattice of $C(X)$. Since $Pf = (\mu, f)\,1$, P is not strictly positive.

Now suppose that $C_T(X)$ is a lattice in $C(X)$ with modulus $\operatorname{mod} f$ for f in $C_T(X)$. By the *lattice boundary* of $C_T(X)$ we shall mean the set

$$\partial_T(X) = \bigcap \{x \in X\colon \operatorname{mod} f(x) = |f(x)|\},$$

where $\bigcap$ is the intersection over all f from $C_T(X)$. The Example above shows that $\partial_T(X)$ need not include the center of the Markov operator. However, we do not know whether $\partial_T(X)$ is always non-empty. On the other hand, we have the following

PROPOSITION. *The lattice boundary of the lattice $C_T(X)$ is a closed invariant set.*

Proof. Let $f \in C_T(X)$ and $g = |f| - \operatorname{mod} f$. Then, clearly, $g \leqslant 0$, $g \in C(X)$, and $Tg \geqslant |Tf| - \operatorname{mod} f = g$. Therefore, the set $\{g = 0\}$ is closed and invariant. By the definition of $\partial_T(X)$, the assertion follows.

We shall show that in the case of s.m.e. Markov operators T, $\partial_T(X)$ is equal to the conservative set of T. Since $C_T(X)$ is a sublattice of $C(X)$ iff $\partial_T(X) = X$, we obtain (for s.m.e. Markov operators) a simple necessary and sufficient condition for $C_T(X)$ to be a sublattice of $C(X)$.

2. Results. The harmonic diffusion discussed in [8] (Example 1) shows an s.m.e. Markov operator such that $C_T(X)$ is not a sublattice of $C(X)$. Indeed, in that example the space $C_T(X)$ consists exactly of the harmonic functions on the unit disc.

LEMMA. *Let T be a Markov operator on $C(X)$. Then the mapping $U\colon f \to (f, \cdot)$ is an order preserving isometry of $C_T(X)$ into $A(P_T(X))$. If, in addition, T is s.m.e., then U is onto.*

Proof. First, we note that U is order preserving. For $f_1 \leqslant f_2$ the inequality $Uf_1 \leqslant Uf_2$ follows from the definition of U. Now, let $Uf \leqslant Ug$, i.e., $(f, \mu) \leqslant (g, \mu)$ for every invariant measure μ. For arbitrary $x \in X$ there exist a subnet (n') and an invariant measure

$$\mu_x = \lim_{n'} A_{n'}^* \delta_x$$

for which $(f, \mu_x) \leqslant (g, \mu_x)$. Hence $f \leqslant g$. The mapping U is an isometry since

$$\|Uf\| = \sup_{\mu \in P_T(X)} |(f, \mu)| \geqslant \sup_{x \in X} |(f, \mu_x)| = \sup_{x \in X} |f(x)| = \|f\|.$$

To see that the mapping is onto whenever T is s.m.e., it suffices, for every $h \in A(P_T(X))$, to define $f(x) = h(P^* \delta_x)$, where P denotes the associated Markov projection. Then we have

$$(f, \mu) = \int h(P^* \delta_x) d\mu(x) = h\left(\int P^* \delta_x d\mu(x)\right) = h(P^* \mu)$$

for an arbitrary probability measure μ on X and

$$Pf(x) = (f, P^* \delta_x) = h(P^* \delta_x) = f(x),$$

so $Pf = f$, $f \in C_T(X)$, and $h = Uf$ (the above integrals are meant in the sense of weak integrals of Pettis).

By the fact that $C_T(X)$ is a lattice and by the above result we see that $A(P_T(X))$ is a lattice. Hence, by the application of the Bauer Theorem we have (see [1], p. 103)

COROLLARY 1 (see also [7]). *If T is s.m.e., then $P_T(X)$ is a Bauer simplex.*

The following theorem gives a necessary and sufficient condition for $C_T(X)$ to be a sublattice in the case of an s.m.e. Markov operator.

THEOREM 1. *Let T be s.m.e. Then $C_T(X)$ is a sublattice of $C(X)$ iff $P^* \delta_x$ is ergodic for every $x \in X$.*

Proof. By the preceding remark, $C_T(X)$ is a lattice and, by Corollary 1, $P_T(X)$ is a Bauer simplex. We take $f \in C_T(X)$ and denote the lattice modulus of f in $C_T(X)$ by $\operatorname{mod} f$.

For the sufficiency we must show that $\operatorname{mod} f = |f|$. Let $x \in X$. Then $\operatorname{mod} f \in C_T(X)$, so it is constant on supports of ergodic measures [7] and

$$\operatorname{mod} f(x) = (\operatorname{mod} f, P^* \delta_x) = U(\operatorname{mod} f)(P^* \delta_x) = |Uf(P^* \delta_x)| = |f(x)|$$

by the isomorphism of $C_T(X)$ with $C(\operatorname{ex} P_T(X))$ (see [1], Theorem II.4.3).

Conversely, if $\mu = P^* \delta_x$ is not ergodic for some $x \in X$, then, also by [1], there exist $h \in A(P_T(X))$ and $f \in C_T(X)$ with $Uf = h$ such that $|h(\mu)| < U(\operatorname{mod} f)(\mu)$. By the definition of measure μ we have $|f(x)| < \operatorname{mod} f(x)$, i.e., $C_T(X)$ is not a sublattice of $C(X)$.

Suppose for the moment that T is an arbitrary Markov operator. As in [7] we denote by $\mathcal{G}$ the partition of X generated by the level sets of $C_T(X)$. Let $\mathcal{E}$ be the collection of those sets of $\mathcal{G}$ which support invariant probabilities. The set $W = \bigcup \{E\colon E \in \mathcal{E}\}$ is called the *conservative set of the Markov operator* T and $E \in \mathcal{E}$ are the ergodic sets. W is always closed (not necessarily invariant), and if T is s.m.e., then each ergodic set is invariant and supports exactly one invariant (ergodic) probability (see [7]). Therefore, for an s.m.e. T we have $W = \{x \in X\colon P^* \delta_x \in \operatorname{ex} P_T(X)\}$.

Remark. From the proof of Theorem 1 we see that if T is s.m.e., then for every $f \in C_T(X)$

$$\operatorname{mod} f(x) = |f(x)| \quad \text{iff} \quad x \in W.$$

COROLLARY 2. *If T is s.m.e., then $\partial_T(X) = W$.*

Throughout the rest of the paper suppose that the Markov operator T has *topologically ergodic decomposition* (t.e.d.), i.e., each ergodic set is invariant (so W is invariant) and supports exactly one invariant probability (see [7]). The second condition means that $C_T(X)$ separates the ergodic measures.

THEOREM 2. *If T has t.e.d., then $C_T(X)$ is a sublattice of $C(X)$ iff the conservative set of T is equal to the whole space.*

Proof. If T has t.e.d., then, by the separation theorem of Iwanik, $A_T(X)$, the closed subalgebra of $C(X)$ generated by $C_T(X)$, separates $P_T(X)$ (see [2], Theorem 4). Hence, if $C_T(X)$ is a sublattice of $C(X)$, then $A_T(X) = C_T(X)$, so T is s.m.e. and Theorem 1 yields $W = X$. The sufficiency is a consequence of Theorem 1 and the fact that the restricted operator $T|_W$ is s.m.e. on W (see [7]).

Added in proof. In a next paper (to appear in Colloquium Mathematicum) we continue to study the lattice properties of $C_T(X)$. By using different methods we have proved, in particular, that the lattice boundary $\partial_T(X)$ is always nonempty.

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