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UNIFORM CONVERGENCE AND DECAY OF FOURIER SERIES ON COMPACT NILMANIFOLDS

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1. If G is a simply connected nilpotent Lie group it admits discrete co-compact subgroups Γ precisely when the Lie algebra $\mathfrak{g}$ has rational structure constants. It is well known that the right action R of G on $L^2(\Gamma\backslash G)$ decomposes into factor representations corresponding to a discrete set $(\Gamma\backslash G)^\wedge \subseteq \widehat{G}$ of irreducibles, and that each $\pi \in (\Gamma \backslash G)^\wedge$ appears with finite multiplicity $m(\pi)$. The spectrum $(\Gamma \backslash G)^\wedge$ and the multiplicities were computed explicitly in terms of coadjoint orbits in $\mathfrak{g}^*$ using polarizations [10], [17], and later investigations have shown that these calculations may be performed entirely in terms of canonical objects associated with orbits [3], [6], [9].

If $P_\pi, \pi \in (\Gamma \backslash G)^\wedge$, are the projections onto the primary components in $L^2(\Gamma \backslash G)$ we have

$$f = \sum_{\pi\in(\Gamma\backslash G)^\wedge} P_\pi f \tag{1}$$

in the L^2 sense and it is natural to ask how much smoothness on f is required to ensure this sum converges uniformly and absolutely. Part of the problem is that continuity of f does not always imply continuity of $P_\pi f$ (see [18]), and a number of papers [1], [2], [3], [14] have been concerned with estimates on the order of P_π — that is, on the number of derivatives f must have for $P_\pi f$ to be continuous. It is known [1] that P_π maps $C^\infty(\Gamma \backslash G)$ into itself; this also follows immediately once one shows that the C^∞ vectors for R are precisely $C^\infty(\Gamma\backslash G)$ (see [4], Appendix 1, or [16]). As for (1), Sobolev estimates were used in [1] to prove that

(2) If $\{P_k\}$ are mutually orthogonal projections such that
(i) P_k commutes with R_g, $\forall g \in G$, $k \in \mathbb{N}$,
(ii) $\sum_{k=1}^{\infty} P_k = I$ in the strong operator sense,

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then $P_k f$ is continuous and $\sum_{k=1}^{\infty} P_k f = f$ in the $\|\cdot\|_\infty$-norm for any $f \in C^{(s)}(\Gamma \setminus G)$ if $s \geq [n/2]+1$.

This is a statement about unconditional convergence since condition (ii) remains valid under any rearrangement; thus the series must be absolutely convergent at each point in $\Gamma \setminus G$.

We are left with some issues that do not seem to be resolvable using only Sobolev or other estimates that make no reference to the geometry of the orbits for $\pi \in (\Gamma \setminus G)^\wedge$. One would like to know whether $\sum_{\pi \in (\Gamma\setminus G)^\wedge} \|P_\pi f\|_\infty$ is finite, and if possible obtain information about the relative size of the terms. By adapting a result from [11] one can obtain estimates of the following kind: if $k > 0$ is sufficiently large then $P_\pi f$ is continuous and there is a $C > 0$ such that

$$(3) \quad \|P_\pi f\|_\infty \leq C\|\pi\|^{-k} \quad \forall \pi \in (\Gamma \setminus G)^\wedge, \quad \pi \not\equiv 1; \quad \forall f \in C^{(k)}(\Gamma \setminus G),$$

where $\mathcal{O}_\pi$ is the $\mathrm{Ad}^* G$-orbit associated with π and

$$\|\pi\| = \text{distance from } \mathcal{O}_\pi \text{ to the origin in } \mathfrak{g}^*.$$

We then show how the exponent should be chosen to ensure that $\sum_\pi \|P_\pi f\|_\infty$ is finite. Our final result is:

THEOREM 1.1. *Let Γ be a discrete cocompact subgroup in a simply connected nilpotent Lie group G and let P_π be the projection onto the π-primary subspace of $L^2(\Gamma \setminus G)$. If*

$$k \geq (n+1) + ([n/2]+1)$$

then $P_n f$ is continuous for every $f \in C^{(k)}(\Gamma \setminus G)$ and

$$\sum_{\pi \in (\Gamma\setminus G)^\wedge} \|P_\pi f\|_\infty < \infty$$

with $\sum_{\pi \in (\Gamma\setminus G)^\wedge} P_\pi f = f$ absolutely and uniformly convergent.

The exponent k can undoubtedly be improved, especially for groups having square integrable representations. We would like to thank Len Richardson for his comments on [1].

2. Proof of the Theorem. From [4], Chapter 5, and [12], we know that there are "lattice subgroups" Γ_0 of finite index in Γ: $\log \Gamma_0$ is a free $\mathbf{Z}$-module in $\mathfrak{g}$ with $n = \dim \mathfrak{g}$ generators. Consider the representations

$$R = \mathrm{ind}(\Gamma \uparrow G, 1), \quad R_0 = \mathrm{ind}(\Gamma_0 \uparrow G, 1), \quad \rho = \mathrm{ind}(\Gamma_0 \uparrow \Gamma, 1);$$

then $\rho \geq 1_\Gamma$ (the trivial representation on Γ) so $R_0 \geq \mathrm{ind}(\Gamma \uparrow G, 1) = R$, and we see that $(\Gamma_0 \setminus G)^\wedge \supseteq (\Gamma \setminus G)^\wedge$. For a lattice subgroup there is a very clean orbital criterion for determining the spectrum:

$$\pi \in (\Gamma_0 \setminus G)^\wedge \Leftrightarrow \text{the } \mathrm{Ad}^* G\text{-orbit } \mathcal{O}_\pi \text{ meets } (\log \Gamma_0)^{\mathbf{Z}} = \{\ell \in \mathfrak{g}^* : \ell(\log \Gamma_0) \subseteq \mathbf{Z}\} \text{ nontrivially} \tag{4}$$

(see [12], Theorem 1). For Γ we only have ($\Rightarrow$).

Fix an inner product in $\mathfrak{g}$ and take the corresponding inner product in $\mathfrak{g}^*$ such that the map $X \to \ell_X$ with $\ell_X(Y) = (Y, X)$ is an isometry. For any $\pi \in \widehat{G}$, with orbit $\mathcal{O}_\pi \subseteq \mathfrak{g}^*$, define $\|\pi\| = \mathrm{dist}(\mathcal{O}_\pi, 0)$ using the metric. If $(\Gamma \setminus G)_0^\wedge$ is the spectrum with the trivial one-dimensional representation excluded, then for any exponent $k > 0$ we have

$$\sum_{\pi' \in (\Gamma_0 \setminus G)_0^\wedge} \|\pi'\|^{-k} \geq \sum_{\pi \in (\Gamma \setminus G)_0^\wedge} \|\pi\|^{-k}. \tag{5}$$

In view of the estimates (3) our result follows if the left-hand side of (5) is finite, and for this we deal only with a lattice subgroup.

The estimate (3) arises from Theorem 4.1 of [11]. If $X_1, \ldots, X_n$ is any orthonormal basis for $\mathfrak{g}$ then $L = (-1)\sum_{i=1}^n X_i^2$ is a symmetric element in the enveloping algebra $\mathfrak{u}(\mathfrak{g})$ and is invariant under any change of orthonormal bases. On $C^\infty(\Gamma \setminus G)$ the operators $R(L)^k$, $k = 1, 2, \ldots$, are essentially self-adjoint because $C^\infty(\Gamma \setminus G)$ is the set $\mathcal{H}_R^\infty$ of C^∞ vectors for R (see [4], Appendix 1, or [16]). In fact, by [13], Theorem 2.2, for any unitary representation π of G, $\pi(L^k)$ is essentially self-adjoint on the space of Gårding vectors $\mathcal{H}_\pi^\gamma = \pi(C_0^\infty(G))\mathcal{H}_\pi$. These, however, are known [7] to coincide with the C^∞ vectors $\mathcal{H}_\pi^\infty$. (Alternatively, slight modifications of [13] make it work for $\mathcal{H}_\pi^\infty$ in place of $\mathcal{H}_\pi^\gamma$.)

For any unitary π we write $\bar{\pi}(L^k)$ for the unique self-adjoint extension, the closure of $(\pi(L^k), \mathcal{H}_\pi^\infty) = (\pi(L)^k, \mathcal{H}_\pi^\infty)$. Then one can show that

$$\bar{\pi}(L^k) = [\bar{\pi}(L)]^k = \int_{-\infty}^{\infty} \lambda^k E(d\lambda)$$

where $\int_{-\infty}^{\infty} \lambda E(d\lambda)$ is the spectral resolution of $\bar{\pi}(L)$ (see [8], pp. 1196–1201). Thus if $\lambda_0 = \min\{\mathrm{sp}\,\bar{\pi}(L)\}$ we have

$$\begin{aligned} \lambda_0^k &= (\min\{\mathrm{sp}\,\bar{\pi}(L)\})^k = \min\{\mathrm{sp}\,\bar{\pi}(L^k)\} \\ &= \inf\{(\bar{\pi}(L^k)u, u) : \|u\| = 1,\ u \in \mathrm{Dom}(\bar{\pi}(L^k))\} \\ &= \inf\{(\pi(L)^k u, u) : \|u\| = 1,\ u \in \mathcal{H}_\pi^\infty\}. \end{aligned} \tag{6}$$

In [11] it is shown that $\lambda_0 \geq \|\pi\|^2$ if π is irreducible, and hence

$$\text{If} \quad \pi \in \widehat{G}, \quad \text{then} \quad \lambda_0^k \geq \|\pi\|^{2k} \quad \forall k \in \mathbf{N}. \tag{7}$$

Let $\pi \in \widehat{G}$ and $\rho = m \cdot \pi$ a primary representation ($m < \infty$). Then $\rho(L^k)$ is essentially self-adjoint in $\mathcal{H}_\rho^\infty$. Any projection Q that intertwines the action of ρ preserves C^∞ vectors, so if we split $\mathcal{H}_\rho = \bigoplus_i \mathcal{H}_{\pi_i}$ with

$\pi_i = \rho \mid \mathcal{H}_{\pi_i} \simeq \pi$ we get $Q_i(\mathcal{H}_\rho^\infty) \subseteq \mathcal{H}_{\pi_i}^\infty$. Hence $\mathcal{H}_\rho^\infty = \bigoplus \mathcal{H}_{\pi_i}^\infty$, from which we easily see that if $\rho = m \cdot \pi$ for some $\pi \in \widehat{G}$ we have

$$\begin{aligned} &\lambda_0^k = \inf\{(\rho(L^k)u, u) : \ \|u\| = 1\,,\ u \in \mathcal{H}_\rho^\infty\} = \min\{\operatorname{sp} \bar{\rho}(L^k)\}\,, \\ &\lambda_0^k \geq \|\rho\|^{2k} \quad \text{if we let } \|\rho\| = \|\pi\|\,. \end{aligned} \tag{8}$$

Now consider a typical π-primary projection P_π on L^2; on range $(P_\pi) = \mathcal{H}_\pi$, R acts like $\rho = m(\pi) \cdot \pi$ for $\pi \in (\Gamma \setminus G)^\wedge$. The C^∞ vectors for R are $C^\infty(\Gamma \setminus G)$; since P_π intertwines the R-action we get $P_\pi(C^\infty) \subseteq C^\infty$. Likewise, we also see that $P_\pi(C^\infty) \subseteq \mathcal{H}_\rho^\infty$. The quadratic form $\phi \mapsto (R(L)^k\phi, \phi)$ on $C^\infty(\Gamma \setminus G)$ can be written as either

$$\|R(L)^\ell \phi\|_0^2 \quad \text{or} \quad \sum_{i=1}^n \|R(X_i)R(L)^\ell\phi\|_0^2 \quad (\|u\|_0 = L^2\text{-norm})$$

according to whether $k = 2\ell$ or $k = 2\ell + 1$, $\ell \in \mathbf{N}$. Thus if we impose the Sobolev norm $\|\phi\|_k^2 = \sum_{|\alpha| \leq k} \|R(X^\alpha)\phi\|_0^2$ on $C^\infty(\Gamma \setminus G)$ we see that there is a $C' > 0$ such that $(R(L)^k\phi, \phi) \leq C'\|\phi\|_k^2$, $\forall \phi \in C^\infty$. Hence

$$\begin{aligned} C'\|\phi\|_k^2 &\geq (R(L)^k\phi, \phi) \geq (R(L)^k P_\pi\phi, P_\pi\phi) \\ &= (\rho(L)^k P_\pi\phi, P_\pi\phi) \quad (R(L) \text{ commutes with } P_\pi \text{ and } I - P_\pi) \\ &\geq \lambda_0^k \|P_\pi\phi\|_0^2 \geq \|\pi\|^{2k}\|P_\pi\phi\|_0^2 \quad (\text{by } (8))\,. \end{aligned}$$

The Sobolev space $\mathcal{H}_k(\Gamma \setminus G) = \{u \in L^2 :\ R(X^\alpha)u \in L^2$ (as a distribution), $\forall |\alpha| \leq k\}$ is complete in the $\|\cdot\|_k$-norm, and is the $\|\cdot\|_k$-norm closure of $C^\infty(\Gamma \setminus G)$ as can be seen by adopting the idea behind [19], pp. 58–59. If $\phi_n \in C^\infty$, $\|\phi_n - f\|_k \to 0$, then $\|\phi_n - f\|_0 \to 0$ and $\|P\phi_n - Pf\|_0 \to 0$, so

$$\|\pi\|^{2k}\|P\phi\|_0^2 \leq C'\|\phi\|_k^2 \quad \forall \phi \in \mathcal{H}_k\,. \tag{9}$$

In particular, this applies to any $\phi \in C^{(k)}(\Gamma \setminus G)$. Finally, notice that if $u \in L^2$ and $R(X^\alpha)u \in L^2$ (as a distribution) then we have

$$P(R(X^\alpha)u) = R(X^\alpha)Pu \quad \text{as distributions}$$

and in particular $R(X^\alpha)Pu \in L^2$. [If $\phi \in C^\infty$ then $(PR(X^\alpha)u, \phi) = (R(X^\alpha)u, P\phi)$ and $P\phi \in C^\infty$, so the latter is equal to $(u, R(X^\alpha)^*P\phi) = (u, PR(X^\alpha)^*\phi) = (R(X^\alpha)Pu, \phi)$.] For $s > 0$, $f \in C^{(k+s)}$, and $|\alpha| \leq s$, we then have $R(X^\alpha)f \in C^{(k)}$, $P(R(X^\alpha)f) = R(X^\alpha)Pf \in L^2$, and

$$\begin{aligned} &\|R(X^\alpha)Pf\|_0 = \|PR(X^\alpha)f\|_0 \leq C'\|\pi\|^{-k}\|R(X^\alpha)f\|_s \\ &\qquad\qquad\qquad\qquad\qquad\quad \leq C'\|\pi\|^{-k}\|f\|_{k+s} \quad \forall |\alpha| \leq s\,, \\ &\|Pf\|_s \leq C'\|\pi\|^{-k}\|f\|_{k+s}\,. \end{aligned} \tag{10}$$

If $s = [n/2] + 1$, standard Sobolev theory says that $f \in C^{(k+s)} \Rightarrow P_\pi f$ is continuous and $\|P_\pi f\|_\infty \leq C\|P_\pi f\|_s \leq C'C\|f\|_{k+s}\|\pi\|^{-k}$, $\forall \pi \in (\Gamma \setminus G)_0^\wedge$, where $C > 0$ is an absolute constant. Thus we get

$$\sum_{\pi\in(\Gamma\backslash G)^\wedge} \|P_\pi f\|_\infty < \infty$$

if we choose the exponent k so that $\sum_{\pi\in(\Gamma\backslash G)^\wedge_0} \|\pi\|^{-k} < \infty$, and for this it suffices to consider the lattice subgroup $\Gamma_0 \subseteq \Gamma$ and choose k so that

$$\sum_{\pi\in(\Gamma_0\backslash G)^\wedge_0} \|\pi\|^{-k} < \infty. \tag{11}$$

Let $Y_1, \ldots, Y_n$ be a strong Malcev basis for $\mathfrak{g}$ strongly based on Γ_0. Then $\operatorname{Ad}\gamma(\Gamma_0) \subseteq \Gamma_0$ for $\gamma \in \Gamma_0$ so $(\log\Gamma_0)^{\mathbf{Z}}$ is $\operatorname{Ad}^*(\Gamma_0)$-invariant. Let $K = \{\exp(s_1Y_1)\ldots\exp(s_nY_n) : 0 \le s_i < 1\}$; then $\Gamma K = G$, so if an orbit $\mathcal{O}_\pi$ meets $(\log\Gamma_0)^{\mathbf{Z}}$ and $\ell \in \mathcal{O}_\pi$, $\operatorname{Ad}^*(K)\ell$ must meet $\mathcal{O}_\pi \cap (\log\Gamma_0)^{\mathbf{Z}}$. As before, $\pi \in (\Gamma_0 \setminus G)^\wedge \Leftrightarrow \mathcal{O}_\pi \cap (\log\Gamma_0)^{\mathbf{Z}} \neq \emptyset$. For $\pi \in (\Gamma_0 \setminus G)^\wedge$ define $|\pi| = \operatorname{dist}(0, \mathcal{O}_\pi \cap (\log\Gamma_0)^{\mathbf{Z}})$. Then $|\pi| \ge \|\pi\|$, but on the other hand the map $\mathfrak{g}^* \times \mathbf{R}^n \to \mathbf{R}^n$,

$$(\ell, s_1, \ldots, s_n) \to \operatorname{Ad}^*(\exp s_1Y_1 \ldots \exp s_nY_n)\ell$$

is polynomial in $\mathbf{R}^n$ and linear in $\ell \in \mathfrak{g}^*$. Take an element $\ell \in \mathcal{O}_\pi$ such that $\|\ell\| = \|\pi\|$. It is clear that there is a constant C such that $\|\operatorname{Ad}^*(x)\ell\| \le C\|\ell\|$ for all $x \in K$, $\ell \in \mathfrak{g}^*$; this ensures that $|\pi| \le C\|\pi\|$. We may therefore replace $\|\pi\|$ by $|\pi|$ in (11). Evidently

$$\sum_{\pi\in(\Gamma_0\backslash G)^\wedge_0} |\pi|^{-k} \le \sum_{f\in(\log\Gamma_0)^{\mathbf{Z}},\ f\neq 0} \|f\|^{-k}.$$

The latter series converges if $k \ge n+1$. This completes the proof of the theorem. ∎

It is clear that we can control convergence of derivatives $R(X^\alpha)(f) = \sum_{\pi\in(\Gamma\backslash G)^\wedge} P_\pi(R(X^\alpha)f)$ for $|\alpha| \le r$ if we require that $f \in C^{(k+r)}$ where k is the exponent in Theorem 1.1.

2.1. EXAMPLE. If $G = \mathbf{R}^n$ and $\Gamma = \mathbf{Z}^n$ then $(\Gamma \setminus G)^\wedge$ consists of the characters χ_λ, $\lambda \in \mathbf{Z}^n$, on the torus $\Gamma \setminus G$. The quasi-regular representation R is multiplicity-free, $m(\pi) = 1$ for all π, and the decomposition $f = \sum_{\pi\in(\Gamma\backslash G)^\wedge} P_\pi f$ is just the Fourier transform $\sum_{\lambda\in\mathbf{Z}^n} \widehat{f}(\lambda)\chi_\lambda$ on $L^2(\Gamma \setminus G)$. To get $\sum_{\lambda\in\mathbf{Z}^n} |\widehat{f}(\lambda)| < \infty$ it suffices to require $f \in C^{(k)}(\Gamma \setminus G)$ with $k \ge n+1$. Our estimate would require $k \ge k_1+k_2 = ([n/2]+1)+(n+1)$. In that estimate we required k_1-fold differentiability to ensure continuity of $P_\pi f$ and control of its norm, $\|P_\pi f\|_\infty \le C\|f\|_{k_1}$; in this example $P_\pi f$ is automatically continuous and $\|P_\pi f\|_\infty \le C\|f\|_0$ so our choice of k_1 is too large. The other exponent is chosen so that (11) holds , $k_2 \ge n+1$. That choice cannot be improved here. ∎

2.2. EXAMPLE. Let G be the three-dimensional Heisenberg group, realized as $\mathbf{R}^3$ with the multiplication law

$$(z,y,x)\cdot(z',y',x') = \left(z+z'+\frac{1}{2}xy', y+y', x+x'\right).$$

This is obtained from the Campbell–Hausdorff formula on identifying $\mathfrak{g} = \mathbf{R}^3$ via a basis Z, Y, X such that $[X,Y] = Z$. Then $\log(z,y,x) = zZ + yY + xX$ and $\Gamma = \mathbf{Z} \times 2\mathbf{Z} \times 2\mathbf{Z}$ is a lattice subgroup. Writing $\ell \in \mathfrak{g}^*$ as $\ell = \dot{z}Z^* + \dot{y}Y^* + \dot{x}X^*$ in the dual basis, the coadjoint action becomes

$$\mathrm{Ad}^*_{\exp(zZ+yY+xX)}(\dot{z},\dot{y},\dot{x}) = (\dot{z}, \dot{y} - x\dot{z}, \dot{x} + y\dot{z})$$

and the $\mathrm{Ad}^* G$ orbits are

$$\begin{aligned} \mathrm{Ad}^* G(\dot{z},0,0) &= \{\dot{z}\} \times \mathbf{R}^2 \quad \text{if } \dot{z} \neq 0, \\ \mathrm{Ad}^* G(0,\dot{y},\dot{x}) &= \{(0,\dot{y},\dot{x})\} \quad \text{if } \dot{z} = 0. \end{aligned}$$

The integral orbits that meet $\log \Gamma$ are given by

$$\begin{aligned} \mathcal{O}_n &= \{n\} \times \mathbf{R}^2, \quad n \neq 0,\ n \in \mathbf{Z}, \\ \mathcal{O}_{p,q} &= \{(0,p,q)\}, \quad p,q \in \frac{1}{2}\mathbf{Z}, \end{aligned}$$

and the corresponding irreducible representations $\pi \in (\Gamma \backslash G)^\wedge$ are precisely those with integral orbit

$$\begin{aligned} &\pi_n \text{ modeled on } L^2(\mathbf{R}) \text{ with } \pi_n(z,0,0) = e^{2\pi i n z} I, \\ &\pi_{p,q} \text{ one-dimensional with } \pi_{p,q}(z,y,x) = e^{2\pi i (py+qx)}. \end{aligned}$$

For the infinite-dimensional π_n there is a single rational ideal $\mathfrak{m} = \mathbf{R}Z + \mathbf{R}Y$ that polarizes all the representatives $\ell_n = nZ^*$ $(n \neq 0)$, and $\mathfrak{m} \cap \log \Gamma = (\mathbf{Z})Z + (2\mathbf{Z})Y$. The integral characters in $\widehat{M}$ and their orbits under G and Γ are easily calculated. The number of Γ-orbits in the G-orbit of $\chi_n(\exp H) = e^{2\pi i n Z^*(H)}$, $H \in \mathfrak{m}$, is the multiplicity of π_n in $L^2(\Gamma \backslash G)$, as in [6]: we find that $m(\pi_n) = |n|$, and similarly $m(\pi_{p,q}) \equiv 1$.

Since all orbits in $\mathfrak{g}^*$ are flat, $P_\pi f$ is continuous if $f \in C^{(0)}(\Gamma \backslash G)$ [18]; however, we might need to require that f be in some Sobolev class k_1 to get control of $\|P_\pi f\|_\infty$ in terms of a Sobolev norm. (Our choice was $k_1 \geq 2$.) The second part of our estimate, which says

$$\sum_{\pi \not\equiv 1} \|P_\pi f\|_\infty \leq C \sum_{\pi \not\equiv 1} \|P_\pi f\|_{k_1} \leq C' \Big(\sum_{\pi \not\equiv 1} \|\pi\|^{-k_2}\Big) \|f\|_{k_1+k_2},$$

estimates

$$\sum_{\pi \not\equiv 1} \|\pi\|^{-k_2} \quad \text{by} \quad \sum_{\ell \in (\log \Gamma)^{\mathbf{Z}},\ \ell \not\equiv 0} \|\ell\|^{-k_2}.$$

Here we are overestimating a term $\|\pi\|^{-k_2}$ by summing over all integral points in the orbit; for the 2-dimensional orbits our choice of k_2 is too large.

Our estimate works if $f \in C^{(6)}(\Gamma \setminus G)$, but we actually have finiteness of the sum

$$\sum_{\pi \neq 1} \|\pi\|^{-k_2} = \sum_{(p,q)\neq(0,0)} (p^2+q^2)^{-k_2/2} + \sum_{n\neq 0} n^{-k_2} \quad . \tag{12}$$

if $k_2 \geq 3$. Thus we only need $f \in C^{(5)}(\Gamma \setminus G)$, and in fact we could reduce k_1 by considering explicit formulas for the projections P_π that are available in the flat-orbit case [6]. ∎

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