COLLOQUIUM MATHEMATICUM

VOL. LVII 1989 FASC. 2

ON THE COEXISTENCE OF VARIOUS GEOMETRIES

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Geometry of a space $\mathfrak{S}$ determines geometry of the space $d(\mathfrak{S})$ of its directions. On the other hand, geometry of $d(\mathfrak{S})$ determines that of $\mathfrak{S}$. This gives a natural structure in the class of various geometries. In this paper we restrict our considerations to four (five) most popular geometries.

Some results concerning groups of automorphisms are given.

1. Classical and full geometries.

1.1. By *n-dimensional Euclidean space* we understand, as usually (up to isomorphism), R^n provided with the standard geometrical linear structure (i.e., the family of all k-hyperplanes for $1 \leqslant k < n$) and with the perpendicularity relation given by the equality

$$\sum_{i=1}^{n} a_i b_i = 0 \tag{1}$$

(i.e., two $(n-1)$-hyperplanes

$$a_0 + \sum_{i=1}^{n} a_i x_i = 0 \quad \text{and} \quad b_0 + \sum_{i=1}^{n} b_i x_i = 0$$

are *perpendicular* iff this equality holds).

Similarly we define

the *Minkowskian n-space* replacing (1) by

$$a_1 b_1 - \sum_{i=2}^{n} a_i b_i = 0, \tag{2}$$

the *Bolyai-Lobatchevskian n-space* replacing R^n by the unit open ball and (1) by

$$a_0 b_0 - \sum_{i=1}^{n} a_i b_i = 0, \tag{3}$$

the *elliptic n-space* replacing R^n by $(R^n - \{(0, \ldots, 0)\})/\sim$ (where $\sim$ denotes the proportionality relation) and (1) by

$$\sum_{i=0}^{n} a_i b_i = 0. \tag{4}$$

Let **Eucl**^{n}, **Min**^{n}, **BL**^{n} and **Ell**^{n} denote corresponding classes of spaces and Eucl^n, Min^n, BL^n and Ell^n their theories (geometries).

1.2. The *Möbius n-space* is understood here as a structure isomorphic to an n-sphere in R^{n+1} with the family of all k-spheres, for $1 \leqslant k < n$, treated as the family of all hyperplanes, and usual Euclidean perpendicularity of spheres.

The class of Möbius n-spaces will be denoted by **Mö**^{n} and its theory (geometry) by Mö^n.

1.3. For our purpose it is convenient to consider full spaces.

Given a space $\mathfrak{S} \in (\mathbf{Eucl}^n \cup \mathbf{Min}^n \cup \mathbf{BL}^n \cup \mathbf{Ell}^n)$, the *full space* $\tilde{\mathfrak{S}}$ is obtained as follows:

First, we extend the geometrical linear structure of $\mathfrak{S}$ to a projective n-space $\mathfrak{P}^n$ over reals so that

(i) $|\mathfrak{S}| \subset |\mathfrak{P}^n|$ (we use the symbol $|\mathfrak{S}|$ for the universe of the space $\mathfrak{S}$);

(ii) for every k-hyperplane X in $\mathfrak{S}$, $1 \leqslant k < n$, there exists a k-hyperplane $\tilde{X}$ in $\mathfrak{P}^n$ such that $X = \tilde{X} \cap |\mathfrak{S}|$.

Second, we extend the perpendicularity in $\mathfrak{S}$ to the minimal symmetric relation satisfying the following condition:

(iii) if $(n-1)$-hyperplanes $X_1, \ldots, X_n$ in $\mathfrak{S}$ are independent, have a common point p, and are perpendicular to some $(n-1)$-hyperplane X, then every $(n-1)$-hyperplane in $\tilde{\mathfrak{S}}$ passing through p is perpendicular to $\tilde{X}$.

The following two remarks are obvious:

1.4. Remark. If $\mathfrak{S} \in \mathbf{Ell}^n$, then $\tilde{\mathfrak{S}} = \mathfrak{S}$.

1.5. Remark. For $\mathfrak{S} \in \mathbf{Mö}^n$ there is no full space.

In the sequel we shall use the following

1.6. PROPOSITION. *The extended perpendicularity in $\tilde{\mathfrak{S}}$ can be described by the same formula as the original one in $\mathfrak{S}$.*

Proof. I. For $\mathfrak{S} \in (\mathbf{Eucl}^n \cup \mathbf{Min}^n)$ the set $|\tilde{\mathfrak{S}}| - |\mathfrak{S}|$ is an $(n-1)$-hyperplane H in $\mathfrak{P}^n$. Let H be described by $x_0 = 0$. By formulae (1) and (2) of 1.1 it suffices to show that H is singular, i.e.,

$$\forall \tilde{X}\ (\tilde{X} \perp H).$$

Let $\tilde{X} \neq H$. Then there exist independent $(n-1)$-hyperplanes $X_1, \ldots, X_n$ in $\mathfrak{S}$ such that $X_i \perp X$ for $i = 1, \ldots, n$. The hyperplanes $\tilde{X}_1, \ldots, \tilde{X}_n$ have a common point in H. By condition (iii) of 1.3 we have $\tilde{X} \perp H$.

Let now $\tilde{X} = H$. Any n independent $(n-1)$-hyperplanes different from H and passing through a point $p \in H$ are perpendicular to H. Then $H \perp H$ by (iii).

II. Let $\mathfrak{S} \in \mathbf{BL}^n$. For any independent $(n-1)$-hyperplanes $X_1, \ldots, X_n$ in $\mathfrak{S}$ perpendicular to some $(n-1)$-hyperplane X, the hyperplanes $\tilde{X}_1, \ldots, \tilde{X}_n$ pass through the pole of $\tilde{X}$ with respect to the $(n-1)$-dimensional hyperquadric given by

$$x_0^2 - \sum_{i=1}^{n} x_i^2 = 0$$

(see (3) of 1.1). This defines perpendicularity for every pair of $(n-1)$-hyperplanes in $\tilde{\mathfrak{S}}$ such that at least one of them contains a hyperplane of $\mathfrak{S}$. Since the perpendicularity in $\tilde{\mathfrak{S}}$ is minimal, there is no more possibility.

The condition "two hyperplanes are perpendicular iff one of them passes through the pole of the other" corresponds to formula (3) of 1.1.

III. For $\mathfrak{S} \in \mathbf{Ell}^n$ see Remark 1.4.

1.7. CONCLUSION. By 1.6 the spaces $\mathfrak{S}$ and $\tilde{\mathfrak{S}}$ determine uniquely each other.

Indeed, if there exists a singular hyperplane H in $\tilde{\mathfrak{S}}$, then $|\mathfrak{S}| = |\tilde{\mathfrak{S}}| - H$; if there exists an $(n-1)$-dimensional hyperquadric in $\tilde{\mathfrak{S}}$ tangent to all isotropic (perpendicular to itself) $(n-1)$-hyperplanes, then $|\mathfrak{S}|$ is the interior of such a hyperquadric; in the remaining cases $|\mathfrak{S}| = |\tilde{\mathfrak{S}}|$.

Therefore, we can investigate $\tilde{\mathfrak{S}}$ instead of $\mathfrak{S}$.

2. Spaces of directions. We are going to define the space $d(\mathfrak{S})$ for a given space $\mathfrak{S}$.

2.1. The set of directions of a space $\mathfrak{S}$, defined as usually, can be described in terms of the full space as follows:

$$|d(\mathfrak{S})| := \mathrm{Cl}_{\tilde{\mathfrak{S}}}|\mathfrak{S}| - |\mathfrak{S}|.$$

2.2 By the *space $d(\mathfrak{S})$ of directions* of an n-space $\mathfrak{S}$ we mean the set $|d(\mathfrak{S})|$ provided with the family of k-hyperplanes, $k = 1, \ldots, n-2$, and the perpendicularity relation which are defined as follows:

k-hyperplanes are nondegenerate intersections of $|d(\mathfrak{S})|$ and $(k+1)$-hyperplanes in $\mathfrak{S}$;

for any two hyperplanes X and Y in $\mathfrak{S}$,

$$\tilde{X} \cap |d(\mathfrak{S})| \perp \tilde{Y} \cap |d(\mathfrak{S})| \leftrightarrow \tilde{X} \perp \tilde{Y}. \tag{5}$$

2.3. Remark. By 1.6 and (5), in any coordinate systems of $\tilde{\mathfrak{S}}$ the perpendicularity in $d(\mathfrak{S})$ can be described by the same formula as the perpendicularity in $\mathfrak{S}$.

2.4. THEOREM. (i) *If* $\mathfrak{S} \in \mathbf{Min}^n$, *then* $d(\mathfrak{S}) \in \mathbf{BL}^{n-1}$.

(ii) *If* $\mathfrak{S} \in \mathbf{BL}^n$, *then* $d(\mathfrak{S}) \in \mathbf{M\ddot{o}}^{n-1}$.

(iii) *If* $\mathfrak{S} \in \mathbf{Eucl}^n$, *then* $d(\mathfrak{S}) \in \mathbf{Ell}^{n-1}$.

Proof. (i) and (iii). We choose a coordinate system in $\tilde{\mathfrak{S}}$ so that $|d(\mathfrak{S})|$ is the hyperplane P_∞: $x_0 = 0$. Now, in view of (1)–(4) of 1.1, it suffices to change coordinates in P_∞:

$$x_i \mapsto x_{i-1} \quad \text{for } i = 1, \ldots, n.$$

(ii) Choose a coordinate system in $\tilde{\mathfrak{S}}$ such that $|d(\mathfrak{S})|$ is the unit sphere. We have to show that any two $(n-2)$-spheres in $|d(\mathfrak{S})|$ are (Euclidean) orthogonal iff $(n-1)$-hyperplanes containing them are conjugate with respect to $|d(\mathfrak{S})|$. We prove this Euclidean theorem.

Let p be a common point of two $(n-2)$-spheres S_1 and S_2 in $|d(\mathfrak{S})|$ and let H be the $(n-1)$-hyperplane tangent to $|d(\mathfrak{S})|$ at p. There exist $(n-1)$-hyperplanes H_1 and H_2 such that

$$S_i = |d(\mathfrak{S})| \cap H_i \quad \text{for } i = 1, 2.$$

Consider three vectors u, v, w: $u \perp H_1$, $v \perp H$, $w \| [\overrightarrow{pb}]$, where b is the pole of H_1 with respect to $|d(\mathfrak{S})|$. Since u, v, w are dependent, $w \perp H \cap H_1$. We have $w \| H$, so

$$H_1 \text{ and } H_2 \text{ are conjugate} \leftrightarrow w \| H_2 \leftrightarrow H \cap H_2 \perp H \cap H_1.$$

2.5. If we have some n-space $\mathfrak{S}$ in $\mathfrak{P}^{n+1}$ placed in such a way that $(n-1)$-hyperplanes in $\mathfrak{S}$ are subsets of $(n-1)$-hyperplanes in $\mathfrak{P}^{n+1}$, then we can treat $\mathfrak{S}$ as the direction space of some $(n+1)$-space $D(\mathfrak{S})$. For this purpose it suffices to define perpendicularity by (5), i.e., two hyperplanes in $D(\mathfrak{S})$ are perpendicular iff they pass through some perpendicular hyperplanes in $\mathfrak{S}$.

2.6. THEOREM. (i) *If* $\mathfrak{S} \in \mathbf{BL}^n$, *then* $D(\mathfrak{S}) \in \mathbf{Min}^{n+1}$.

(ii) *If* $\mathfrak{S} \in \mathbf{M\ddot{o}}^n$, *then* $D(\mathfrak{S}) \in \mathbf{BL}^{n+1}$.

(iii) *If* $\mathfrak{S} \in \mathbf{Ell}^n$, *then* $D(\mathfrak{S}) \in \mathbf{Eucl}^{n+1}$.

Proof. (i) and (iii). Let the hyperplane $|\tilde{\mathfrak{S}}|$ in $\mathfrak{P}^{n+1}$ be given by $x_0 = 0$. Then the perpendicularity in $D(\mathfrak{S})$ is given by formulae (3) and (4), respectively, where x_i is replaced by x_{i+1} for $i = 1, \ldots, n$. Thus we obtain formula (2) in the case (i), and (1) in the case (iii), both for $(n+1)$-spaces.

(ii) According to the Euclidean theorem used in the proof of 2.4 (ii), two n-hyperplanes in $D(\mathfrak{S})$ are perpendicular iff they are conjugate with respect to the unit sphere.

3. Relationships between various geometries.

3.1. Classes $\mathbf{M\ddot{o}}^n$ and $\mathbf{Eucl}^n$ are equivalent in the following sense:

By adding one point to a Euclidean space and all Euclidean spheres to the class of hyperplanes and by extending the perpendicularity relation we obtain a Möbius space. More precisely, the additional point belongs to all Euclidean hyperplanes and the extended perpendicularity is Euclidean.

Conversely, by removing from a Möbius space one point and all hyperplanes not passing through it, we obtain a Euclidean space.

Geometrically, this is realized by the stereographic projection (see, e.g., [1]).

3.2. Among all full spaces only elliptic spaces are homogeneous (this is a result of Hamel; see [3], Section 4).

In this sense the n-dimensional elliptic geometry determines uniquely the $(n-1)$-dimensional elliptic geometry (as the geometry of arbitrary $(n-1)$-hyperplane of the elliptic n-space) and determines uniquely the $(n+1)$-dimensional elliptic geometry (as the geometry of the unique homogeneous $(n+1)$-dimensional extension of the elliptic n-space).

3.3. By 2.4, 2.6, 3.1 and 3.2 we have the following diagram:

$$\begin{array}{ccccccccc}
 & \mathrm{Min}^n & & & \mathrm{Min}^5 & & \mathrm{Min}^4 & & \\
 & | & & & | & & | & & \\
 & \mathrm{BL}^{n-1} & & & \mathrm{BL}^4 & & \mathrm{BL}^3 & & \\
 & | & & & | & & | & & \\
 & \mathrm{M\ddot{o}}^{n-2} & \sim & \mathrm{Eucl}^{n-2} & \mathrm{M\ddot{o}}^3 & \sim \mathrm{Eucl}^3 & \mathrm{M\ddot{o}}^2 & \sim & \mathrm{Eucl}^2 \\
 & & & | & & | & & & | \\
\ldots & \text{———} & & \mathrm{Ell}^{n-3} \text{—} \ldots \text{——} & & \mathrm{Ell}^2 & \text{———} & & \mathrm{Ell}^1
\end{array}$$

In the diagram any vertical motion corresponds to passing from geometry of a space to geometry of its space of directions or conversely, and any horizontal motion is illustrated by 3.1 or 3.2.

3.4. According to the diagram the most economical geometrical description of all geometries discussed can be reduced to the following two theories:

metric geometry of one-dimensional Euclidean circle,

n-dimensional projective geometry.

Indeed, the first of them coincides with Ell^1. The diagram contains a simple recipe how any of the geometries investigated can be obtained.

3.5. The diagram proposes a special point of view of the old question: what is the geometry of real physical space. It shows that any choice of a geometry as a tool to describe some phenomenon determines the class of phenomena which can be described by another (arbitrarily chosen) geometry.

Let us consider, e.g., the Landau–Pomerantchuk paradox of the electron scattering. The electron scattering is a phenomenon in the space-time (Min^4). Since the regeneration of the self-field of the electron depends (up to physical constants) on directions of this space-time only, it should be described in BL^3.

In [2] Feinberg tried to give such a description without geometrical arguments.

4. Groups of automorphisms.

4.1. By the *dilatation group* $Dil(\mathfrak{S})$ of a space $\mathfrak{S}$ we understand the subgroup of the group $Aut(\mathfrak{S})$ of automorphisms of $\mathfrak{S}$, which consists of all transformations preserving every direction.

4.2. Let D^n be the dilatation group of an affine n-space over reals. It consists of all homotheties and translations.

Using the diagram (more precisely, using 2.4) we obtain

4.3. THEOREM. (i) $Aut(\mathbf{Min}^n)/D^n = Aut(\mathbf{BL}^{n-1})$.

(ii) $Aut(\mathbf{BL}^n) = Aut(\mathbf{M\ddot{o}}^{n-1})$.

(iii) $Aut(\mathbf{Eucl}^n)/D^n = Aut(\mathbf{Ell}^{n-1})$.

Indeed, it suffices to observe that

$$Dil(\mathbf{Min}^n) = Dil(\mathbf{Eucl}^n) = D^n \quad \text{and} \quad Dil(\mathbf{BL}^n) = \{\mathrm{Id}\}.$$

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Reçu par la Rédaction le 7.7.1987;
en version modifiée le 11.11.1987