COLLOQUIUM MATHEMATICUM

XLII DÉDIÉ À LA MÉMOIRE D'EDWARD MARCZEWSKI 1979

EXTENSIONS OF POSITIVE OPERATORS AND EXTREME POINTS. I

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The paper falls into two independent sections.

The first section is concerned with extensions of positive operators and functionals on ordered vector spaces. It has been suggested in part by a paper by Łoś and Marczewski [3] on extensions of finitely additive positive measures. The interested reader can easily apply our results to set functions (cf. the part of Section 2 preceding Lemma 2).

In the second section we present a characterization of the extreme points of the set of all positive extensions of a given operator which generalizes a theorem of Douglas [1]. As an application a simplified proof of a generalization of a result due to Plachky [4] is given.

Throughout the paper we adhere to the terminology of Schaefer's monograph [5]. We use the following notation. Y stands for an order complete vector lattice over the reals R. If $A \subset Y$ is not majorized (minorized), we write $\sup A = \infty$ (respectively, $\inf A = -\infty$). We adopt the convention that $\sup\emptyset = -\infty$ and $\inf\emptyset = \infty$. Throughout M stands for an arbitrary vector subspace of an ordered real vector space X. The space of all (positive) linear operators from M into Y is denoted by $L(M, Y)$ (respectively, $L_+(M, Y)$). Given a vector subspace N of X with $M \subset N$ and $T \in L_+(M, Y)$, we put

$$E(T, N) = \{S \in L_+(N, Y)\colon\ S|M = T\}.$$

The notation $E(T, X)$ is abbreviated to $E(T)$. Finally, for $T \in L_+(M, Y)$ we denote by $U(T)$ the domain of uniqueness of T, i.e.

$$U(T) = \{x \in X\colon\ S_1(x) = S_2(x) \text{ for all } S_1, S_2 \in E(T)\}.$$

Clearly, $U(T)$ is a vector subspace of X containing M.

1. Extensions of positive operators and functionals. With every $T \in L_+(M, Y)$ we associate two maps $T_i, T_e: X \to Y \cup \{\pm\infty\}$ defined by

$$\left.\begin{aligned} T_i(x) &= \sup\{T(z)\colon x \geqslant z \in M\} \\ T_e(x) &= \inf\{T(z)\colon x \leqslant z \in M\} \end{aligned}\right\} \quad \text{for all } x \in X.$$

Obviously, for every $S \in E(T)$ and $x \in X$ we have $T_i(x) \leqslant S(x) \leqslant T_e(x)$. This suggests the problem of the existence for a given $x_0 \in X$ and $y_0 \in [T_i(x_0), T_e(x_0)]$ of $S \in E(T)$ with $S(x_0) = y_0$ (cf. [3], p. 269). It will be of our primary concern in this section. First of all we note some more or less known results.

(i) Given $T \in L_+(M, Y)$, $x_0 \in X$ and $y_0 \in Y$ with $y_0 \in [T_i(x_0), T_e(x_0)]$, there exists a unique $S \in E(T, \operatorname{lin}(M \cup \{x_0\}))$ such that $S(x_0) = y_0$.

Clearly, the operator S defined, for $z \in M$ and $t \in R$, by $S(z+tx_0) = T(z)+ty_0$ is as desired.

Using the Kuratowski-Zorn lemma, we get, by (i), the following result which is due essentially to L. V. Kantorovič (cf. [6], Theorem X.3.1). This theorem forms a first solution of the problem posed above.

THEOREM 1. *Suppose that M is a majorizing (i.e. cofinal) subspace of X. Then, given $T \in L_+(M, Y)$, $x_0 \in X$ and $y_0 \in Y$ with $y_0 \in [T_i(x_0), T_e(x_0)]$, there exists $S \in E(T)$ such that $S(x_0) = y_0$. In particular, $U(T) = \{x \in X\colon T_i(x) = T_e(x)\}$.*

In the remaining part of this section we assume additionally that X has a (strong) order unit u. Then

(ii) Given $T \in L_+(M, Y)$ and $y_0 \in Y$ with $y_0 \in [T_i(u), T_e(u)]$, there exists $S \in E(T)$ such that $S(u) = y_0$. In particular, $E(T) \neq \emptyset$ if and only if $T_i(u) < \infty$.

This follows easily from (i) and Theorem 1 as $\operatorname{lin}(M \cup \{u\})$ majorizes X.

LEMMA 1. *Suppose $T \in L_+(M, Y)$ and $T_i(u) < \infty$. Given $x_0 \in X$ and $y_0 \in Y$ with $y_0 \in [T_i(x_0+\varepsilon u), T_e(x_0-\varepsilon u)]$ for some $\varepsilon \in R_+$, $\varepsilon > 0$, there exists $S \in E(T)$ such that $S(x_0) = y_0$.*

Proof. By (i), there exists $S_0 \in E(T, \operatorname{lin}(M \cup \{x_0\}))$ with $S_0(x_0) = y_0$. Therefore, in view of (ii), it is enough to prove that $S_{0i}(u) < \infty$, i.e. the set $\{T(z)+ty_0\colon z \in M,\ t \in R \text{ and } z+tx_0 \leqslant u\}$ is majorized. To this end observe first that $T_e(x_0-t^{-1}u) \geqslant y_0$ whenever $t \geqslant \varepsilon^{-1}$. If, moreover, $z+tx_0 \leqslant u$, then $-t^{-1}z \geqslant x_0-t^{-1}u$, so that $T(-t^{-1}z) \geqslant y_0$. Hence $T(z)+ +ty_0 \leqslant 0$. Analogously, the same holds if $z+tx_0 \leqslant u$ and $t \leqslant -\varepsilon^{-1}$. Finally, take $n \in R_+$ with $tx_0 \leqslant nu$ for all $|t| \leqslant \varepsilon^{-1}$. If $z+tx_0 \leqslant u$, then

$$T(z) \leqslant T_i(u-tx_0) \leqslant (n+1)T_i(u),$$

which yields the assertion.

For $T \in L_+(M, Y)$ and $x \in X$ we put

$$T_i(x+) = \inf\{T_i(x+tu):\ t \in R_+,\ t > 0\},$$
$$T_e(x-) = \sup\{T_e(x-tu):\ t \in R_+,\ t > 0\}.$$

With this notation we have

(iii) Given $T \in L_+(M, Y)$, for any $S \in E(T)$ and $x \in X$ we have $T_i(x) \leqslant T_i(x+) \leqslant S(x) \leqslant T_e(x-) \leqslant T_e(x)$.

Now we restrict our attention to the case $Y = R$.

As

$$]T_i(x+), T_e(x-)[= \bigcup_{\varepsilon>0} [T_i(x+\varepsilon u), T_e(x-\varepsilon u)],$$

we get, by Lemma 1 and (iii), the following result:

THEOREM 2. *Suppose $T \in L_+(M, R)$ and $T_i(u) < \infty$. Then, given $x_0 \in X$ and $y_0 \in]T_i(x_0+), T_e(x_0-)[$, there exists $S \in E(T)$ such that $S(x_0) = y_0$. In particular, $U(T) = \{x \in X:\ T_i(x+) = T_e(x-)\}$.*

Let us note that the assumption that $y_0 \in]T_i(x_0+), T_e(x_0-)[$ in Theorem 2 cannot be weakened to $y_0 \in [T_i(x_0+), T_e(x_0-)]$ even in case $T_i(x_0+), T_e(x_0-) \in R$. This is illustrated by the following

Example. Let $N = \{1, 2, \ldots\}$ and put

$$X = \Big\{\sum_{j=1}^{n} t_j 1_{A_j}:\ t_j \in R,\ A_j \subset N\Big\} \quad \text{and} \quad u = 1_N.$$

Arrange the prime numbers into a sequence $\{p_n:\ n = 2, 3, \ldots\}$ (without repetitions) and put

$$D_n = \{mp_n:\ m \in N\} \quad \text{for } n = 2, 3, \ldots$$

and

$$D_1 = N \setminus \{p_n:\ n = 2, 3, \ldots\}.$$

Consider the subspace $M = \big\{\sum_{j=1}^{n} t_j 1_{D_j}:\ t_j \in R\big\}$ of X and $T\colon M \to R$ given by the formula

$$T\Big(\sum_{j=1}^{n} t_j 1_{D_j}\Big) = \sum_{j=1}^{n} t_j j^{-1}.$$

Clearly, $T \in L_+(M, R)$ and $T_i(1_N) = 1$. Put

$$D = \bigcup_{\substack{n,m=2\\ n\neq m}}^{\infty} D_n \cap D_m \quad \text{and} \quad x_0 = 1_D.$$

Obviously, $T_e(1_D-) = 1$. Observe that

$$T_i(1_D + n^{-1}1_N) = n^{-1}\sum_{j=1}^{n+1} j^{-1},$$

so that $T_i(1_D+) = 0$. Finally, we show that there is no $S \in E(T)$ with $S(1_D) = 0$. Otherwise, as

$$\sum_{j=2}^{n+1} 1_{D_j} - n1_D \leqslant 1_N,$$

we have

$$\sum_{j=2}^{n+1} j^{-1} \leqslant S(1_N),$$

so that $S(1_N) = \infty$, a contradiction.

2. Extreme points. We shall give a characterization of the extreme positive extensions of a given operator $T \in L_+(M, Y)$. This characterization can be viewed as a generalization of a result of Douglas ([1], Theorem 1).

THEOREM 3. *Suppose X is a vector lattice and $S \in E(T)$. Then $S \in \operatorname{extr} E(T)$ if and only if $\inf\{S(|x-z|)\colon z \in M\} = 0$ for each $x \in X$.*

Proof. The "if" part. In order to prove that $S \in \operatorname{extr} E(T)$ it is enough to show that for any $Q \in E(T)$ and $t \in R_+$ with $tS - Q$ positive we have $S = Q$. Clearly,

$$|Q(x)-Q(z)| \leqslant Q(|x-z|) \leqslant tS(|x-z|).$$

Hence for $z \in M$ and $x \in X$ we get

$$|S(x)-Q(x)| \leqslant |S(x)-S(z)| + |Q(x)-Q(z)| \leqslant (1+t)S(|x-z|).$$

It follows that $S(x) = Q(x)$ for $x \in X$.

The "only if" part. Put $P(x) = \inf\{S(|x-z|)\colon z \in M\}$ for $x \in X$. Then P is a Y_+-valued seminorm on X with the properties: $P(z) = 0$ for $z \in M$ and $P(x) \leqslant S(|x|)$ for $x \in X$. If $P \neq 0$, then, by a generalized version of the Hahn-Banach theorem ([5], II.7.9), there would exist $S_0 \in L(X, Y)$ such that $S_0 \neq 0$ and $|S_0(x)| \leqslant P(x)$ for $x \in X$. Hence, by the above-stated properties of P, $S \pm S_0 \in E(T)$, so that $S \notin \operatorname{extr} E(T)$.

We shall apply this result to set functions. First let us fix some more notation. In the remaining part of the paper $\mathscr{D}$ stands for a (non-empty) family of subsets of a set Ω. By $\mathscr{D}_r$ we denote the ring generated by $\mathscr{D}$, and by $\mathscr{R}$ a ring of subsets of Ω containing $\mathscr{D}$. Put

$$s(\mathscr{D}) = \Big\{\sum_{j=1}^{n} t_j 1_{D_j}\colon\ t_j \in R,\ D_j \in \mathscr{D}\Big\}.$$

Clearly, $s(\mathscr{D})$ is an ordered real vector space. Moreover,

(iv) If $\mathscr{D}$ is closed under finite intersections, then $s(\mathscr{D})$ is lattice-ordered and $s(\mathscr{D}) = s(\mathscr{D}_r)$.

Definition. A function $\mu\colon \mathscr{D} \to Y_+$ is a *quasi-content* provided that it extends to an additive $\nu\colon \mathscr{D}_r \to Y_+$.

With each quasi-content μ on $\mathscr{D}$ we associate a $T_\mu \in L_+(s(\mathscr{D}), Y)$ defined by

$$T_\mu\Big(\sum_{j=1}^{n} t_j 1_{D_j}\Big) = \sum_{j=1}^{n} t_j \mu(D_j), \qquad \text{where } t_j \in R,\ D_j \in \mathscr{D},$$

and the set $E(\mu)$ of all quasi-contents on $\mathscr{R}$ extending μ. Then

(v) The mapping $\nu \to T_\nu$ is an affine isomorphism of the convex sets $E(\mu)$ and $E(T_\mu)$.

LEMMA 2. *If ν is a quasi-content on $\mathscr{R}$ and $\mathscr{S}$ is a ring with $\mathscr{S} \subset \mathscr{R}$, then for each $A \in \mathscr{R}$ we have*

$$\inf\{T_\nu(|1_A - g|)\colon g \in s(\mathscr{S})\} = \inf\{\nu(A\Delta B)\colon B \in \mathscr{S}\}.$$

Proof. Suppose that $B_j \in \mathscr{S}$ are pairwise disjoint. Then

$$\begin{aligned} T_\nu\Big(\Big|1_A - \sum_{j=1}^{n} t_j 1_{B_j}\Big|\Big) &= \nu\Big(A \setminus \bigcup_{j=1}^{n} B_j\Big) + \sum_{j=1}^{n} |1-t_j|\,\nu(A\cap B_j) + \sum_{j=1}^{n} |t_j|\,\nu(B_j \setminus A) \\ &\geqslant \nu\Big(A\setminus \bigcup_{j=1}^{n} B_j\Big) + \sum_{j=1}^{n} \inf\{\nu(B_j\setminus A),\ \nu(A\cap B_j)\}. \end{aligned}$$

Hence (cf. [5], p. 50, (1′))

$$\begin{aligned} &T_\nu\Big(\Big|1_A - \sum_{j=1}^{n} t_j 1_{B_j}\Big|\Big) \\ &\geqslant \nu\Big(A\setminus \bigcup_{j=1}^{n} B_j\Big) + \inf\Big\{\sum_{k\in I} \nu(B_k \setminus A) + \sum_{l\in\{1,\ldots,n\}\setminus I} \nu(A\cap B_l)\colon I \subset \{1,\ldots,n\}\Big\} \\ &= \inf\Big\{\nu\Big(A\Delta \bigcup_{k\in I} B_k\Big)\colon I \subset \{1,\ldots,n\}\Big\} \end{aligned}$$

which yields the assertion.

Now we are in a position to prove the announced application of Theorem 3. The result we obtain is a generalization of a theorem due to Plachky ([4], Theorem 1). The denseness condition appearing in it has been also studied by Lipecki ([2], Section 3).

THEOREM 4. *Suppose μ is a quasi-content on $\mathscr{D}$. Then*

(a) *If $\nu \in \operatorname{extr} E(\mu)$, then $\inf\{\nu(A\Delta C)\colon C \in \mathscr{D}_r\} = 0$ for each $A \in \mathscr{R}$.*

(b) *If $\mathscr{D}$ is closed under finite intersections, $\nu \in E(\mu)$ and $\inf\{\nu(A\Delta C)\colon C \in \mathscr{D}_r\} = 0$ for each $A \in \mathscr{R}$, then $\nu \in \operatorname{extr} E(\mu)$.*

Proof. (a) By assumption and (v), $T_\nu \in \operatorname{extr} E(T_\mu)$. Hence, in view of Theorem 3 and (iv), $\inf\{T_\nu(|1_A - g|)\colon g \in s(\mathscr{D}_r)\} = 0$ for each $A \in \mathscr{R}$. Applying Lemma 2 with $\mathscr{S} = \mathscr{D}_r$, we get the assertion.

(b) By assumption and (iv), $\inf\{T_\nu(|f-g|)\colon g \in s(\mathscr{D})\} = 0$ for each $f \in s(\mathscr{R})$. An application of Theorem 3 and (v) completes the proof.

Postscript. After the original version of the paper had been prepared the authors learned about a work by C. Portenier containing a result which covers our Theorem 3 for $Y = R$ (*Points extrémaux et densité*, Mathematische Annalen 209 (1974), p. 83-89, Théorème 3.5).

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Reçu par la Rédaction le 21. 2. 1978