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REMARKS ON SYMMETRICAL OPERATIONS

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We adopt the terminology and notation of [1] and [2]. In particular, two abstract algebras $(A; \boldsymbol{F}_1)$ and $(A; \boldsymbol{F}_2)$ having the same class of algebraic operations will be treated here as identical. Given an algebra $\mathfrak{A}$, we denote by $\mathscr{S}(\mathfrak{A})$ the set of all non-negative integers n for which there exists an n-ary non-trivial algebraic operation in $\mathfrak{A}$ depending on every variable. The investigation of sets $\mathscr{S}(\mathfrak{A})$ was suggested by E. Marczewski. In particular, he proved that for algebras without algebraic constants and with a k-ary symmetrical (or even quasi-symmetrical) algebraic operation the set $\mathscr{S}(\mathfrak{A})$ contains the arithmetical progression $k+j(k-1)$ $(j = 0, 1, \ldots)$ (see [2]). This Theorem for $k = 2$ was previously obtained by J. Płonka in [4]. Further results for quasi-symmetrical operations are contained in [6]. Moreover, a complete description of the sets $\mathscr{S}(\mathfrak{A})$ for idempotent algebras was given in [5]. Obviously, for unary algebras, i. e. for algebras with unary fundamental operations, the set $\mathscr{S}(\mathfrak{A})$ is contained in $\{0, 1\}$. Therefore from now on *we shall consider only non-unary algebras*, i. e. algebras for which $\mathscr{S}(\mathfrak{A}) \cap \{2, 3, \ldots\} \neq \emptyset$.

Algebras with symmetrical fundamental operations will be called *symmetrical*. Moreover, algebras in which all algebraic operations depending on every variable are symmetrical will be called *completely symmetrical*. The present note contains a complete description of the sets $\mathscr{S}(\mathfrak{A})$ for symmetrical and completely symmetrical algebras which solves a problem raised by E. Marczewski in [2] for symmetrical operations.

First we shall give some examples of the sets $\mathscr{S}(\mathfrak{A})$ for symmetrical algebras.

1. Let A be the set of all non-negative integers. Put $f_0(x) = 0$ and, for $n \geqslant 1$, $f_n(x_1, x_2, \ldots, x_n) = 1$ if $x_1, x_2, \ldots, x_n$ is a permutation of $2, 3, \ldots, n+1$ and $f_n(x_1, x_2, \ldots, x_n) = 0$ in the opposite case. For any set E of non-negative integers such that $0 \in E$ and $E \cap \{2, 3, \ldots\} \neq \emptyset$ we put $\boldsymbol{F}_E = \{f_n\colon n \in E\}$ and $\mathfrak{A}_E = (A; \boldsymbol{F}_E)$. It is very easy to verify that $\mathscr{S}(\mathfrak{A}_E) = E$ and $\mathfrak{A}_E$ is a completely symmetrical algebra with 0 as an algebraic constant. On the other hand, each completely symmetrical algebra with algebraic constants satisfies the conditions $0 \in \mathscr{S}(\mathfrak{A})$ and $\mathscr{S}(\mathfrak{A}) \cap \{2, 3, \ldots\} \neq \emptyset$. Thus *the necessary and sufficient condition for a set E of non-negative integers to be a set $\mathscr{S}(\mathfrak{A})$ for a completely symmetrical algebra with algebraic constants is $0 \in E$ and $E \cap \{2, 3, \ldots\} \neq \emptyset$.*

2. Let $\mathfrak{A}$ be an at least two-element commutative semigroup of idempotents. The algebra $\mathfrak{A}$ is completely symmetrical, has no self-dependent elements and $\mathscr{S}(\mathfrak{A}) = \{2, 3, \ldots\}$.

3. Let G be an at least two-element Boolean group. For any subgroup H of G we denote by $\boldsymbol{F}_H$ the family of ternary operations of the form $x+y+z+a$, where $a \in H$, and we put $\mathfrak{A}_H = (G; \boldsymbol{F}_H)$. The algebra $\mathfrak{A}_H$ is completely symmetrical, has no self-dependent elements, and $\mathscr{S}(\mathfrak{A}_H) = \{3, 5, \ldots\}$ if $H = \{0\}$ or $\mathscr{S}(\mathfrak{A}_H) = \{1, 3, 5, \ldots\}$ if $H \neq \{0\}$.

4. Let A be the class of all subsets of the set $\{1, 2, \ldots, m\}$ $(m \geqslant 3)$, $a = \{1, 2\}$ and $\mathfrak{A} = (A; a \cap x \cap y)$. The algebra $\mathfrak{A}$ is completely symmetrical, has no algebraic constants and $\mathscr{S}(\mathfrak{A}) = \{1, 2, \ldots\}$.

5. Let $\mathfrak{A}_H$ be the algebra defined in Example 3 over an infinite Boolean group G. Let $a_1, a_2, \ldots$ be a sequence of elements of G such that for each r the elements $a_1, a_2, \ldots, a_r$ do not belong to a subalgebra of A_H generated by less than r elements. For any even positive integer r we put

$$g_r(x_1, x_2, \ldots, x_{r+1}) = a_1,$$

if $x_1, x_2, \ldots, x_{r+1}$ is a permutation of the system $a_1, a_1, a_2, \ldots, a_r$ and

$$g_r(x_1, x_2, \ldots, x_{r+1}) = \sum_{j=1}^{r+1} x_j$$

in the opposite case. Put $\mathfrak{B}_H = (G; \boldsymbol{F}_H \cup \{g_r\})$. The algebra $\mathfrak{B}_H$ is symmetrical. One can prove that the algebras $\mathfrak{A}_H$ and $\mathfrak{B}_H$ have the same $(r-1)$-ary algebraic operations. Consequently, the algebra $\mathfrak{B}_H$ has no self-dependent elements. Moreover, the $(r+2n)$-ary operation

$$f(x_1, x_2, \ldots, x_{r+2n}) = g_r(x_1, x_2, \ldots, x_r) + \sum_{j=r+1}^{r+2n} x_j,$$

where $n \geqslant 0$, is algebraic in $\mathfrak{B}_H$ and depends on every variable. Thus $\mathscr{S}(\mathfrak{B}_H) = \{3, 5, \ldots\} \cup \{r, r+1, \ldots\}$ if $H = \{0\}$ and $\mathscr{S}(\mathfrak{B}_H) = \{1, 3, 5, \ldots\} \cup \{r, r+1, \ldots\}$ if $H \neq \{0\}$.

6. Let G be an at least two-element Boolean group and N the set of all positive integers. We define a mapping h from $G \cup N$ onto G by the formulas $h(x) = x$ in G and $h(x) = 0$ in N. Further, for any even positive integer r we define an $(r+1)$-ary operation h_r in $G \cup N$ by the *conditions:*

$$h_r(x_1, x_2, \ldots, x_{r+1}) = 1$$

if $x_1, x_2, \ldots, x_{r+1}$ is a permutation of $2, 2, 3, \ldots, r+1$ and

$$h_r(x_1, x_2, \ldots, x_{r+1}) = \sum_{j=1}^{r+1} h(x_j)$$

in the opposite case. Given an arbitrary non-void set Q of even positive integers, we put $\boldsymbol{F}_Q = \{h_r\colon r \in Q\}$ and $\mathfrak{A}_Q = (G \cup N; \boldsymbol{F}_Q)$. Of course, the algebra $\mathfrak{A}_Q$ is symmetrical and its subalgebra $(G; \boldsymbol{F}_Q)$ is equal to $(G; x+y+z)$. Thus $\mathscr{S}(\mathfrak{A}_Q) \supset \{3, 5, \ldots\}$. Further, $h_r(x, x, \ldots, x) = h(x)$ for $r \in Q$ and, consequently, h is the only unary non-trivial algebraic operation in $\mathfrak{A}_Q$. Hence it follows that $\mathfrak{A}_Q$ has no algebraic constants and $1 \in \mathscr{S}(\mathfrak{A}_Q)$. Since the operation $h_r(x_1, x_1, x_2, \ldots, x_r)$ depends on every variable, we have the inclusion $\mathscr{S}(\mathfrak{A}_Q) \supset Q$. Moreover, for each algebraic n-ary operation f in $\mathfrak{A}_Q$ the composition $h\big(f(x_1, x_2, \ldots, x_n)\big)$ is a sum of an odd number of operations $h(x_1), h(x_2), \ldots, h(x_n)$ and, for all $r \in Q$, $h_r(x_1, x_2, \ldots, x_{r+1}) \in G \cup \{1\}$. Hence and from the definition of fundamental operations h_r it follows that every n-ary algebraic operation g, where n is an even integer and $n \notin Q$, is a sum of an odd number of operations $h(x_1), h(x_2), \ldots, h(x_n)$ and, consequently, does not depend on every variable. Thus $\mathscr{S}(\mathfrak{A}_Q) = \{1, 3, 5, \ldots\} \cup Q$.

The purpose of this note is to prove that above quoted examples give all possible sets $\mathscr{S}(\mathfrak{A})$ for symmetrical and completely symmetrical algebras. In the sequel $\{m, m+1, \ldots\}$ for $m = \infty$ will denote the empty set.

THEOREM 1. *A set E of non-negative integers is a set $\mathscr{S}(\mathfrak{A})$ for a symmetrical algebra without algebraic constants if and only if either $E \supset \{1, 3, 5, \ldots\}$ and $0 \notin E$ or $E = \{3, 5, \ldots\} \cup \{m, m+1, \ldots\}$, where $2 \leqslant m \leqslant \infty$.*

THEOREM 2. *A set E is a set $\mathscr{S}(\mathfrak{A})$ for a symmetrical algebra without self-dependent elements if and only if either $E = \{1, 3, 5, \ldots\} \cup \{m, m+1, \ldots\}$ or $E = \{3, 5, \ldots\} \cup \{m, m+1, \ldots\}$, where $2 \leqslant m \leqslant \infty$.*

THEOREM 3. *A set E is a set $\mathscr{S}(\mathfrak{A})$ for a completely symmetrical algebra without algebraic constants if and only if $E = \{1, 2, 3, \ldots\}$, $E = \{2, 3, \ldots\}$, $E = \{1, 3, 5, \ldots\}$, or $E = \{3, 5, \ldots\}$.*

THEOREM 4. *A set E is a set $\mathscr{S}(\mathfrak{A})$ for a completely symmetrical algebra without self-dependent elements if and only if $E = \{2, 3, \ldots\}$, $E = \{1, 3, 5, \ldots\}$, or $E = \{3, 5, \ldots\}$. Moreover, $\mathscr{S}(\mathfrak{A}) = \{2, 3, \ldots\}$ if and only if $\mathfrak{A}$ is an at least two-element commutative semigroup of idempotents. $\mathscr{S}(\mathfrak{A}) \subset \{1, 3, 5, \ldots\}$ if and only if $\mathfrak{A}$ is an algebra $\mathfrak{A}_H$ defined in Example 3 over a Boolean group.*

Before proving the Theorems we shall prove some Lemmas. The smallest integer $k \geqslant 2$ for which there exists a k-ary symmetrical algebraic operation in the algebra $\mathfrak{A}$ will be called *the symmetry index* of $\mathfrak{A}$.

LEMMA 1. *Let $\mathfrak{A}$ be a symmetrical algebra without algebraic constants and with symmetry index $k \geqslant 3$. Let h be a symmetrical k-ary algebraic operation and $2 \leqslant s \leqslant k-1$. Put*

$$h_1(x_1, x_2, \ldots, x_{s+1}) = h(x_1, x_2, \ldots, x_s, x_{s+1}, x_{s+1}, \ldots, x_{s+1}),$$

$$h_2(x_1, x_2, \ldots, x_{s+1}) = h(u_1, u_2, \ldots, u_k),$$

where $u_j = x_{s+1}$ if $1 \leqslant j \leqslant m$, $u_j = x_i$ if $(i-1)n+m < j \leqslant in+m$ $(i = 1, 2, \ldots, s)$ and $k = ns+m$, $1 \leqslant m < s$. Then for any unary algebraic operation g at least one operation $h_1\big(x_1, x_2, \ldots, x_s, g(x_{s+1})\big)$ or $h_2\big(x_1, x_2, \ldots, x_s, g(x_{s+1})\big)$ depends on every variable.

Proof. Both operations $h_1\big(x_1, x_2, \ldots, x_s, g(x_{s+1})\big)$ and $h_2\big(x_1, x_2, \ldots, x_s, g(x_{s+1})\big)$ are symmetrical with respect to $x_1, x_2, \ldots, x_s$. Since $2 \leqslant s < k$, they depend on the variable x_{s+1}. Moreover, to prove the Lemma it suffices to prove that at least one of them depends on a variable x_j $(1 \leqslant j \leqslant s)$. Suppose the contrary

$$h_1\big(x_1, x_2, \ldots, x_s, g(x_{s+1})\big) = f_1(x_{s+1}),$$

$$h_2\big(x_1, x_2, \ldots, x_s, g(x_{s+1})\big) = f_2(x_{s+1}).$$

Setting $v_j = g(x)$ $(1 \leqslant j \leqslant m)$, $v_j = g(y)$ $(m < j \leqslant s)$, we have for all x and y the equation

$$f_1(y) = h_1\big(v_1, v_2, \ldots, v_s, g(y)\big) = h_2\big(g(y), g(y), \ldots, g(y), g(x)\big) = f_2(x),$$

which implies that the algebraic operations f_1 and f_2 are constant. But this contradicts the assumption. The Lemma is thus proved.

For the definition of simple and complete iterations see [2]. The same reasoning as in the proving of Theorem 2 in [2] yields the following Lemma:

LEMMA 2. *Let g_1 and g_2 be the j-th complete and simple iterations of a k-ary symmetrical operation respectively. If g_0 is an $(s+1)$-ary operation and the composition $g_0\big(y_1, y_2, \ldots, y_s, g_1(x_1, x_2, \ldots, x_{k^j+1})\big)$ depends on every variable, then $g_0\big(y_1, y_2, \ldots, y_s, g_2(x_1, x_2, \ldots, x_{k+j(k-1)})\big)$ depends on every variable too.*

LEMMA 3. *If $\mathfrak{A}$ is a symmetrical algebra without algebraic constants and with symmetry index $k \geqslant 3$, then each integer $n \geqslant 3$ different from $k+j(k-1)+1$ $(j = 0, 1, \ldots)$ belongs to $\mathscr{S}(\mathfrak{A})$.*

Proof. From Lemma 1 it follows that all integers n satisfying the inequality $3 \leqslant n \leqslant k$ belong to $\mathscr{S}(\mathfrak{A})$. Further, each integer $n > k$ and different from $k+j(k-1)+1$ $(j = 0, 1, \ldots)$ can be written in the form

$n = k+j(k-1)+s$, where $j \geqslant 0$ and $2 \leqslant s \leqslant k-1$. Let g_1 and g_2 be the j-th complete and simple iterations of a k-ary symmetrical algebraic operation. Put $g(x) = g_1(x, x, \ldots, x)$. By Lemma 1 there exists an $(s+1)$-ary algebraic operation h_0 such that the composition $h_0\big(y_1, y_2, \ldots, y_s, g(y_{s+1})\big)$ depends on every variable. Hence it follows that the composition $h_0\big(y_1, y_2, \ldots, y_s, g_1(x_1, x_2, \ldots, x_{k^{j+1}})\big)$ depends on $y_1, y_2, \ldots, y_s$ and on at least one x_i $(1 \leqslant i \leqslant k^{j+1})$. Since this composition is quasi-symmetrical with respect to variables $x_1, x_2, \ldots, x_{k^{j+1}}$, we infer that it depends on every variable (for the definition of quasi-symmetry see [2]). Hence and from Lemma 2 it follows that the n-ary algebraic operation $h_0\big(y_1, y_2, \ldots, y_s, g_2(x_1, x_2, \ldots, x_{k+j(k-1)})\big)$ depends on every variable. Thus $n \in \mathscr{S}(\mathfrak{A})$ which completes the proof.

Proof of Theorem 1. The sufficiency of the conditions is proved by Examples 3, 5 and 6. Suppose that $\mathfrak{A}$ is a symmetrical algebra without algebraic constants. Since the symmetry index k of $\mathfrak{A}$ is a prime, we infer by Lemma 3 that in the case $k \geqslant 3$ the set $\mathscr{S}(\mathfrak{A})$ contains all odd integers greater than 1. By quoted above Płonka's Theorem this inclusion is also true in the case $k = 2$. Obviously, $0 \notin \mathscr{S}(\mathfrak{A})$. Suppose that $1 \notin \mathscr{S}(\mathfrak{A})$, i. e. A is an idempotent algebra (see [5]). Since $\mathscr{S}(\mathfrak{A}) \supset \{3, 5, \ldots\}$, we infer by Theorem 1 in [5] that either $\mathscr{S}(\mathfrak{A})$ satisfies the assertion of the Theorem or $\mathscr{S}(\mathfrak{A}) = \{2, 3, \ldots, n\} \cup \{n+2, n+3, \ldots\}$, where n is an odd integer. To prove the Theorem it suffices to show that the last case never holds. By Theorem 2 (Assertion 2) in [5] all n-ary algebraic operations in $\mathfrak{A}$ are algebraic in a diagonal algebra and, consequently, are not symmetrical. Thus $k > n$ and, by Lemma 3, $n+1 \in \mathscr{S}(\mathfrak{A})$ which gives the contradiction. The Theorem is thus proved.

LEMMA 4. *Let $\mathfrak{A}$ be a symmetrical algebra without self-dependent elements and without binary algebraic operations depending on every variable. If the set $\mathscr{S}(\mathfrak{A})$ does not contain $\{3, 4, \ldots\}$, then for every ternary algebraic operation f satisfying the conditions*

$$(1) \quad f(x, y, y) = f(y, x, y), \quad f(x, x, x) = f(x, y, y) \quad \textit{or} \quad f(x, x, y),$$

the equations

$$(2) \qquad f(x, y, y) = f(y, y, x) = g(x), \quad g\big(g(x)\big) = x$$

hold.

Proof. Contrary to this let us suppose that there exists an algebraic operation f satisfying (1) and not satisfying (2). Consequently, one of the following three cases holds:

$$(3) \qquad f(x, y, y) = f(y, y, x) = g(y),$$

$$(4) \qquad f(x, y, y) = g(x), \quad f(y, y, x) = g(y),$$

$$(5) \qquad f(x, y, y) = f(y, y, x) = g(x), \quad g\big(g(x)\big) \neq x.$$

If $g(g(x)) = x$, then, by (1) and (3) or (4), for any pair a, b of elements of A the algebra $(a, b; g(f(x, y, z)))$ is isomorphic to one of the Post algebras $\mathfrak{P}^*$ or $\mathfrak{P}$ (see [3], p. 200) and, consequently, $\mathscr{S}(\mathfrak{A}) \supset \{3, 4, \ldots\}$. Thus

$$g(g(x)) \neq x \quad (x \in A), \tag{6}$$

because $\mathfrak{A}$ has no self-dependent elements.

Consider the case (3). If $f(x, y, g(y))$ does not depend on x, then, in view of (1), the equation

$$g(y) = f(y, y, g(y)) = f(g(y), y, g(y)) = f(y, g(y), g(y)) = g(g(y))$$

holds which contradicts (6). Thus $f(x, y, g(y)) = g(x)$, because $\mathfrak{A}$ has no binary algebraic operations depending on every variable. Setting $f_0(x, y, z) = f(x, y, g(z))$ we obtain an operation satisfying (1), (4) and (6). Thus the case (3) can be reduced to the case (4).

Now consider the cases (4) and (5). Suppose that $f(y, g(y), x)$ does not depend on x. Then, by (1) and (4) or (5), we have the equation

$$g(g(y)) = f(g(y), y, y) = f(y, g(y), y) = f(y, g(y), g(y)) = g(y)$$

which contradicts (6). Further, if $f(y, g(y), x)$ does not depend on y, then, by (1) and (4) or (5), $f(y, g(y), x) = g(g(x))$ and, consequently, $g(g(g(y))) = f(y, g(y), g(y)) = g(y)$ which contradicts (6). The Lemma is thus proved.

Proof of Theorem 2. The sufficiency of the conditions of the Theorem is proved by Examples 2, 3 and 5. In order to prove the necessity let us assume that $\mathfrak{A}$ is a symmetrical algebra without self-dependent elements. Moreover, by quoted above Płonka's Theorem, we may assume that the symmetry index k of $\mathfrak{A}$ is greater than 2.

Let h be a symmetrical k-ary algebraic operation and h_j its j-th complete iteration. First let us assume that for each integer j there exists a binary algebraic operation p_j such that the composition $p_j(y_1, h_j(x_1, x_2, \ldots, x_{k^j+1}))$ depends on every variable. Hence it follows, in virtue of Lemma 2, that $k+j(k-1)+1 \in \mathscr{S}(\mathfrak{A})$. Consequently, by Lemma 3, $\mathscr{S}(\mathfrak{A}) \supset \{3, 4, \ldots\}$ which implies the assertion of the Theorem.

Now let us assume that there exists an index j for which the composition $p(y_1, h_j(x_1, x_2, \ldots, x_{k^j+1}))$ with every binary algebraic operation p does not depend on every variable. Since h_j is a quasi-symmetrical operation, the last composition does not depend either on y_1 or on all variables $x_1, x_2, \ldots, x_{k^j+1}$. Consequently, for every binary algebraic operation p the composition $p(x, q(y))$, where $q(x) = h_j(x, x, \ldots, x)$, depends on at most one variable x or y. Put

$$w(x_1, x_2, \ldots, x_k) = h(q(x_1), q(x_2), \ldots, q(x_k)).$$

Of course, the algebra $\mathfrak{A}_0 = (A; w)$ is symmetrical and has no self-dependent elements. Moreover, for any binary algebraic operation p_0 in $\mathfrak{A}_0$ there exists a binary algebraic operation p in $\mathfrak{A}$ such that $p_0(x, y) = p(q(x), q(y))$. Thus there are no binary algebraic operations in $\mathfrak{A}_0$ depending on every variable. Setting $f_1(x, y, z) = w(x, y, z, z, \ldots, z)$ and $f_2(x, y, z) = w(z, x, y, x, y, \ldots, x, y)$ we obtain algebraic operations in $\mathfrak{A}_0$ symmetrical with respect to x and y because the number k of variables in w being the symmetry index of $\mathfrak{A}$ is an odd number. Moreover, $f_1(x, y, y) = f_2(y, y, x)$ which implies that at least one operation $f_1(x, y, y)$ or $f_2(x, x, y)$ does not depend on the variable y and, consequently, either $f_1(x, y, y) = f_1(x, x, x)$ or $f_2(x, x, y) = f_2(x, x, x)$. Thus we have proved the existence of an algebraic ternary operation f in $\mathfrak{A}_0$ satisfying the following conditions:

$$f(x, y, z) = f(y, x, z), \quad f(x, x, x) = f(x, y, y) \text{ or } f(x, x, y).$$

Hence, by Lemma 4, either $\mathscr{S}(\mathfrak{A}_0) \supset \{3, 4, \ldots\}$ or $f(x, y, y) = f(y, x, y) = f(y, y, x) = g(x)$, where $g(g(x)) = x$. In the first case we have the inclusion $\mathscr{S}(\mathfrak{A}) \supset \{3, 4, \ldots\}$ which implies the assertion of the Theorem. In the remaining case setting $f_0(x, y, z) = g(f(x, y, z))$ we get an operation in $\mathfrak{A}$ satisfying the condition

$$f_0(x, y, y) = f_0(y, x, y) = f_0(y, y, x) = x. \tag{7}$$

Let g_n be an n-ary algebraic operation depending on every variable. Put

$$v(x_1, x_2, \ldots, x_{n+2}) = f_0(x_{n+1}, x_{n+2}, g_n(x_1, x_2, \ldots, x_n)).$$

Since, by (7),

$$v(x_1, x_2, \ldots, x_n, x, x) = g_n(x_1, x_2, \ldots, x_n),$$
$$v(x_1, x_2, \ldots, x_n, x_{n+1}, g_n(x_1, x_2, \ldots, x_n)) = x_{n+1},$$
$$v(x_1, x_2, \ldots, x_n, g_n(x_1, x_2, \ldots, x_n), x_{n+2}) = x_{n+2},$$

the operation v depends on every variable. Thus $n+2 \in \mathscr{S}(\mathfrak{A})$ whenever $n \in \mathscr{S}(\mathfrak{A})$. Hence it follows by Theorem 1 that the set $\mathscr{S}(\mathfrak{A})$ satisfies the assertion of the Theorem which completes the proof.

Proof of Theorem 3. The sufficiency of the conditions of the Theorem is proved by Examples 2, 3 and 4. Suppose now that $\mathfrak{A}$ is a completely symmetrical algebra without algebraic constants. If $\mathscr{S}(\mathfrak{A}) \subset \{1, 3, 5, \ldots\}$, then the Theorem is a consequence of Theorem 1. In the opposite case $\mathscr{S}(\mathfrak{A})$ contains an even integer and, consequently, the symmetry index of $\mathfrak{A}$ is equal to 2. In this case the Theorem is a direct consequence of Płonka's Theorem which completes the proof.

Proof of Theorem 4. Examples 2 and 3 show that the conditions of the Theorem are sufficient. In order to prove the necessity of these conditions consider a completely symmetrical algebra $\mathfrak{A}$ without self-dependent elements.

First let us assume that $2 \in \mathscr{S}(\mathfrak{A})$ and, consequently, by Theorem 3, that $\mathscr{S}(\mathfrak{A}) \supset \{2, 3, \ldots\}$. Given an arbitrary binary algebraic operation g depending on every variable, we put $xy = g(x, y)$. Since g is symmetrical, we have the commutative law $xy = yx$. Moreover, the composition $g\big(x, g(y, z)\big)$ depends on every variable and, consequently, is symmetrical. Hence we get the associative law $x(yz) = (xy)z$. Thus xy is a commutative semigroup multiplication in A. Further, the operation $g\big(x, g(y, y)\big)$ depends on both variables and, consequently, is symmetrical. Thus $g\big(x, g(y, y)\big) = g\big(y, g(x, x)\big)$, i. e. $xy^2 = yx^2$. Setting $y = x^2$ into the last formula we obtain $x^5 = x^4$ and, consequently, $(x^4)^2 = x^4$. Hence we get the formula $x^2 = x$ for all elements $x \in A$, because the algebra A has no self-dependent elements. Thus $(A; xy)$ is a commutative semigroup of idempotents. Moreover, $g(x, x) = x$ for every binary algebraic operation depending on every variable. Since for every unary algebraic operation f the composition $f(xy)$ depends on both variables, we have the equation $x = f(x^2) = f(x)$. Thus all unary algebraic operations in $\mathfrak{A}$ are trivial and, consequently, $\mathscr{S}(\mathfrak{A}) = \{2, 3, \ldots\}$. To prove the Theorem in the case $2 \in \mathscr{S}(\mathfrak{A})$ it remains to prove the formula $\mathfrak{A} = (A; xy)$. Let f be an n-ary algebraic operation in $\mathfrak{A}$ depending on every variable and, consequently, symmetrical. We shall prove the formula

$$f(x_1, x_2, \ldots, x_n) = f(x_1 x_j, x_2, \ldots, x_n) \quad (1 \leqslant j \leqslant n). \tag{8}$$

If $f(x_1 x_{n+1}, x_2, \ldots, x_n)$ does not depend on x_{n+1}, then (8) is obvious, Suppose that it depends on x_{n+1} and, consequently, depends on every variable. Then $f(x_1 x_{n+1}, x_2, \ldots, x_n)$ is a symmetrical operation which implies the equation

$$f(x_1 x_{n+1}, x_2, \ldots, x_n) = f(x_1, x_2, \ldots, x_{j-1}, x_j x_{n+1}, x_{j+1}, \ldots, x_n).$$

Setting $x_{n+1} = x_j$ into this equation we obtain formula (8). The iteration of (8) yields the formula

$$f(x_1, x_2, \ldots, x_n) = f\Big(\prod_{j=1}^{n} x_j, \prod_{j=1}^{n} x_j, \ldots, \prod_{j=1}^{n} x_j\Big) = \prod_{j=1}^{n} x_j,$$

which shows that f is algebraic in $(A; xy)$. Thus $\mathfrak{A} = (A; xy)$ which completes the proof in the case $2 \in \mathscr{S}(\mathfrak{A})$.

Now suppose that $2 \notin \mathscr{S}(\mathfrak{A})$, i. e., by Theorem 3, that $\mathscr{S}(\mathfrak{A}) = \{3, 5, \ldots\}$ or $\{1, 3, 5, \ldots\}$. By Theorem 2 (Assertion 3) in [5] $\mathscr{S}(\mathfrak{A}) = \{3, 5, \ldots\}$ if and only if $\mathfrak{A} = (G; x+y+z)$, where G is an at least two-element Boolean group.

Consider the case $\mathscr{S}(\mathfrak{A}) = \{1, 3, 5, \ldots\}$. For each ternary algebraic operation h depending on every variable and, consequently, symmetrical, we have the equation $h(x, x, x) = h(x, y, y)$ or $h(x, x, y)$. Hence in view of Lemma 4 we get the formula $g\big(g(x)\big) = x$, where $g(x) = h(x, x, x)$. Since for every unary algebraic operation f the composition $f\Big(g\big(h(x, y, z)\big)\Big)$ depends on every variable and $f\Big(g\big(h(x, x, x)\big)\Big) = f(x)$, we have the equation $f\big(f(x)\big) = x$. Let $\tilde{A}$ be the class of all algebraic operations g satisfying the condition $g(x, x, \ldots, x) = x$. Each algebraic operation v is a composition of an operation from $\tilde{A}$ and a unary algebraic operation. In fact, $v(x_1, x_2, \ldots, x_n) = f\big(g(x_1, x_2, \ldots, x_n)\big)$, where $f(x) = v(x, x, \ldots, x)$ and $g(x_1, x_2, \ldots, x_n) = f\big(v(x_1, x_2, \ldots, x_n)\big)$. Setting $\mathfrak{A}_0 = (A; \tilde{A})$ we get a completely symmetrical algebra with $\mathscr{S}(\mathfrak{A}_0) = \{3, 5, \ldots\}$ By the previous part of the proof, $\mathfrak{A}_0 = (G; x+y+z)$, where G is an at least two-element Boolean group. Moreover, for any unary algebraic operation f in $\mathfrak{A}$ the operation $f(x)+f(y)+z$ is algebraic in $\mathfrak{A}_0$ and depends on every variable. Thus $f(x)+f(y)+z = x+y+z$, whence the formula $f(x) = x+f(0)$ follows. Let H be the subset of G consisting of all elements $f(0)$, where f are unary algebraic operations in $\mathfrak{A}$. Obviously, H is a subgroup of G and each algebraic operation in A is a composition of operations $x+y+z+a$ $(a \in H)$ and trivial operations. Thus the algebra $\mathfrak{A}$ is equal to the algebra $\mathfrak{A}_H$ defined in Example 3 which completes the proof.

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