COLLOQUIUM MATHEMATICUM

VOL. XXXIX 1978 FASC. 2

REFINED MORLEY NUMBERS

BY

D. W. H. GILLAM (ROCKHAMPTON, QUEENSLAND)

Suppose that L is an uncountable first-order language. In this paper the effects on the value of the Morley number for L of restrictions on the cardinality of the type, or set of types, to be omitted are considered. Specifically, it will be shown, for example, that if a cardinal can be characterized by a theory T and a type Σ, both of power λ, then that cardinal can be characterized by a theory T' and a type Σ', there T' has power λ and Σ' has power equal to the supremum of the inaccessible cardinals smaller than or equal to λ.

Morley numbers (also called *Hanf numbers* for omitting types) were first introduced by Morley in [4], p. 265-273. Basic results and definitions may be found there, or in [5].

1. Introduction. Throughout this paper, ξ and ζ will denote ordinals, and the remaining lower case Greek letters will denote cardinals. For a set A, $|A|$ denotes the cardinality of A. The cardinal $2(\varkappa, \xi)$ is defined by induction on ξ:

$$2(\varkappa, 0) = \varkappa, \quad 2(\varkappa, \xi+1) = 2^{2(\varkappa,\xi)},$$

and if ξ is a limit ordinal, then

$$2(\varkappa, \xi) = \sup\{2(\varkappa, \zeta): \zeta < \xi\}.$$

L denotes a first-order language, T a theory in L, and Σ a *type*, that is, a set of formulae of L. The formulae of Σ are assumed to have only the variable v_0 free. Notation, except where otherwise indicated, is that of [1].

Definition 1.1. The cardinal λ is *characterized by the pair* (T, Σ) if every model of T which omits Σ has power less than λ and, for every $\nu < \lambda$, T has a model of power ν which omits Σ.

From this definition we obtain the following lemma:

LEMMA 1.1. *If* (T, Σ) *characterizes* ν *and* T *is of power at most* $\varkappa$, Σ *is of power* $\lambda \leqslant \varkappa$, *then there is a pair* (T', Σ'), *in an extension of* L, *which* *characterizes ν, such that T' has power at most $\varkappa$ and*

$$\Sigma' = \{v_0 \neq c_\xi\colon\ \xi < \lambda\} \cup \{U(v_0)\}$$

for some set of constants $\{c_\xi\colon\ \xi < \lambda\}$ and some unary relation symbol U.

Proof. Let T and Σ be given such that (T, Σ) characterizes ν, T has power of at most $\varkappa$, and Σ has power λ, where $\lambda \leqslant \varkappa$. If $\nu \leqslant \lambda$, then with the set of new constants $\{c_\xi\colon\ \xi < \lambda\}$ and a new unary relation U adjoined to the language L of T to yield a language L' one takes the theory T' to consist of the sentences:

$$(\forall v_0)\, U(v_0), \qquad \neg c_\xi = c_\zeta \text{ for } \xi, \zeta < \nu \qquad \text{and} \qquad c_\xi = c_0 \text{ for } \nu \leqslant \xi < \lambda.$$

Then (T', Σ') characterizes ν (and reference to T is not required). Thus, it may be assumed that $\nu > \lambda$.

Now, let $\Sigma = \{f_\xi\colon\ \xi < \lambda\}$, and let R be a new binary relation symbol which is adjoined to the language L' described above to yield a language L''. Then T' is the theory $T^{(\neg U)}$ (T relativized to $\neg U$) together with the following sentences:

(1) $U(c_\xi)$ for $\xi < \lambda$,

(2) $(\forall v_0)\big(\neg U(v_0) \to (\exists v_1)\big)\big(R(v_0, v_1) \,\&\, U(v_1)\big)$,

(3) $(\forall v_0)\big(\neg f_\xi(v_0) \leftrightarrow R(v_0, c_\xi)\big)$ for $\xi < \lambda$.

So, for each model B of T, a model A of T' may be constructed such that the reduct of $A \restriction \neg U$ to L is isomorphic to B. In addition, if B omits Σ, then A omits Σ'; and, furthermore, in this case, $|B| \geqslant \lambda$ implies $|B| = |A|$. So, for each $\alpha < \nu$, T' has a model of power α which omits Σ'.

Lastly, to show that (T', Σ') characterizes ν, suppose that A is a model of T' which omits Σ' and $|A| \geqslant \nu$. Then $A \restriction U$ has power at most λ, and since $\nu > \lambda$, $A \restriction \neg U$ must have power greater than or equal to ν. By (2), every element of $A \restriction \neg U$ is related by R to c_ξ for some $\xi < \lambda$ (Σ being omitted by A). So, by (4), for all $a \in |A \restriction \neg U|$, $A \models \neg f_\xi[a]$ for some $\xi < \lambda$. Thus $A \restriction \neg U$ omits Σ and, therefore, has power less than ν — a contradiction. It follows that all models of T' which omit Σ' have power less than ν, and hence (T', Σ') characterizes ν.

Hence, we may assume that Σ is a set of formulae of the form

$$\{\neg(v_0 = c_\xi)\colon\ \xi < \lambda\} \cup \{U(v_0)\},$$

where $\{c_\xi\colon\ \xi < \lambda\}$ is a set of constant symbols and U is a unary relation symbol.

LEMMA 1.2. *If the cardinal λ is characterized by the pair (T, Σ), then or every cardinal ν which is less than λ there is a cardinal ϱ such that $\nu \leqslant \varrho < \lambda$ and T has a model of power ϱ which omits Σ and which has no elementary extension omitting Σ.*

Proof. Suppose that the result does not hold for some cardinal λ and some pair (T, Σ). Then there is a cardinal $\nu < \lambda$ such that every model of T of power greater than or equal to ν which omits Σ has an elementary extension which omits Σ.

So an elementary chain $\{A_\xi: \xi < \lambda\}$ of models of T which omit Σ may be constructed as follows. Let A_0 be such a model of power ν. If A_ξ has been constructed for $\xi < \zeta$, where $\zeta < \lambda$, then let A_ζ be an elementary extension of $\bigcup_{\xi<\zeta} A_\xi$ which omits Σ in case

$$|\bigcup_{\xi<\zeta} A_\xi| < \lambda,$$

and let A_ζ be

$$\bigcup_{\xi<\zeta} A_\xi \quad \text{if } |\bigcup_{\xi<\zeta} A_\xi| = \lambda.$$

Then $\bigcup_{\xi<\lambda} A_\xi$ is a model of T which omits Σ and it has power λ — a contradiction to the fact that λ is characterized by (T, Σ).

If ν is a cardinal characterized by a pair (T, Σ), where T and Σ are contained in a language of power at most $\varkappa$ and $|\Sigma| \leqslant \lambda$, then we say that ν *is characterized at* $(\varkappa, \lambda)$ (by (T, Σ)).

Note that ν is characterized at $(\varkappa, \varkappa)$ if and only if ν is characterized at $(\varkappa, \lambda)$ for every cardinal which is greater than or equal to $\varkappa$.

2. Omitting a single type. To study the problem stated at the beginning of this paper we formulate the following definition:

Definition 2.1. For cardinals $\varkappa$ and λ,

$$\mu_1(\varkappa, \lambda) = \sup\{\nu : \nu \text{ is characterized at } (\alpha, \beta) \text{ for some } \alpha < \varkappa \text{ and } \beta < \lambda\}.$$

This is one of the *refined Morley numbers* referred to in the title of this paper.

Morley [4], p. 265-273, showed that

$$\mu_1(\aleph_1, \aleph_1) = 2(\aleph_0, \aleph_1)$$

and, for any $\varkappa$,

$$\mu_1(\varkappa^+, \varkappa^+) < 2\big(\aleph_0, (2^\varkappa)^+\big).$$

Chang [2] has given a simple proof that $\mu_1(\varkappa^+, \varkappa^+) > 2(\aleph_0, \varkappa^+)$ if $\mathrm{cf}(\varkappa) > \aleph_0$ (this was first observed, assuming $V = L$, by Morley and Morley [6]) and Helling [3], using the generalized continuum hypothesis, showed that if $\mathrm{cf}(\varkappa) = \aleph_0$, then $\mu_1(\varkappa^+, \varkappa^+) = 2(\aleph_0, \varkappa^+)$.

THEOREM 2.1. *For cardinals $\nu, \varkappa, \lambda_1$, and λ, if ν is characterized at $(\varkappa, \lambda_1)$ and, for some λ_2 greater than λ_1 and less than or equal to ν and $\varkappa$, λ_2^+ is characterized at $(\varkappa, \lambda)$, then ν is characterized at $(\varkappa, \lambda)$.*

Proof. First note that $\lambda_1 \leqslant \varkappa$.

Suppose that (T_1, Σ_1) characterizes ν at $(\varkappa, \lambda_1)$ and (T_2, Σ_2) characterizes λ_2^+ at $(\varkappa, \lambda)$ for some λ_2 such that $\lambda_1 \leqslant \lambda_2 \leqslant \varkappa$, where T_1, T_2, and Σ_1, Σ_2 are formulated in languages L_1 and L_2, respectively, which have only logical symbols and individual variables in common. Using Lemma 1.1, we may suppose that

$$\Sigma_1 = \{v_0 \neq c_\xi : \xi < \lambda_3\} \cup \{U(v_0)\}$$

for some set of constant symbols $\{c_\xi : \xi < \lambda_3\}$, where $\lambda_3 \leqslant \lambda_1$, and some unary relation symbol U.

Put

$$L = L_1 \cup L_2 \cup \{U_1, U_2, R\} \cup \{d_\xi : \xi < \lambda_2\},$$

where U_1 and U_2 are new unary relation symbols, R is a new binary relation symbol, and $\{d_\xi : \xi < \lambda_2\}$ is a set of new constant symbols. We consider the case $\lambda_3 < \lambda_2$; the case $\lambda_3 = \lambda_2$ follows from a simplification of the following argument.

Select a model B of T_2 which omits Σ_2, has power λ_2, and has no elementary extension omitting Σ_2 — such a model exists by Lemma 1.2. Let T be a set of sentences in L such that if $A \models T$, then

(1) U_1 and U_2 partition A;

(2) $A \upharpoonright U_i \models T_i$ for $i = 1, 2$;

(3) $A \models U_2(d_\xi)$ for all $\xi < \lambda_2$;

(4) the reduct of $A \upharpoonright \{d_\xi : \xi < \lambda_2\}$ to L_2 is isomorphic to B;

(5) $A \models R(c_\xi, d_\xi)$ for $\xi < \lambda_3$ and $A \models R(c_0, d_\xi)$ for $\lambda_3 \leqslant \xi < \lambda_2$.

Lastly, put $\Sigma = \Sigma_2 \cup \{U_2(v_0)\}$.

Then $|T| \leqslant \varkappa$ and $|\Sigma| \leqslant \lambda$. If C is a model of T which omits Σ, then C omits

$$\{v_0 \neq d_\xi : \xi < \lambda_1\} \cup \{U_2(v_0)\} = \Sigma_3.$$

Indeed, if C realized Σ_3, then the reduct of $C \upharpoonright U_2$ to L_2 would be isomorphic to a strict extension of B by (2), (3), and (4), and so, by the definition of B, it would realize Σ_2. Hence C would realize Σ by the definition of Σ.

It follows that $\Sigma_1 \cup \{U_1(v_0)\} = \Sigma_4$ is omitted by C — for suppose $a \in C$ and a realizes Σ_4 in C. By (5),

$$C \models (\forall v_0)(\exists ! v_1) R(v_1, v_0),$$

where $\exists !$ is interpreted as "there is a unique ...". So there is an element $b \in C$ such that $C \models R(a, b)$ and, by (5), $b \neq d_\xi$ for all $\xi < \lambda_2$. Hence $C \upharpoonright U_2$ realizes Σ_2, and so C realizes Σ, which is a contradiction.

Since $C \restriction U_1$ is a model of T_1 and omits Σ_1, we have $|C \restriction U_1| < \nu$, and since $C \restriction U_2$ is a model of T_2 and omits Σ_2, we obtain $|C \restriction U_2| \leqslant \lambda_2$. Consequently, $|C| < \nu$ (for $\lambda_2 < \nu$).

COROLLARY 2.1. *If ν is characterized at $(\varkappa, \lambda_1)$ and, for some λ_2 greater than or equal to λ_1 and at most equal to ν and $\varkappa$, λ_2 is characterized at $(\varkappa, \lambda)$, then ν is characterized at $(\varkappa, \lambda)$.*

Proof. If λ_2 is characterized at $(\varkappa, \lambda)$, then, using results of Morley [4], p. 265-273, so is λ_2^+, and the result follows from Theorem 2.1.

If λ is a cardinal, put

$$\lambda^{\#} = \sup\{\nu \leqslant \lambda : \nu \text{ is weakly inaccessible}\}$$

and

$$\lambda^{\#\#} = \sup\{\nu \leqslant \lambda : \nu \text{ is strongly inaccessible}\}.$$

THEOREM 2.2. *If ν is characterized at $(\varkappa, \lambda)$, then ν is characterized at $(\varkappa, \lambda^{\#})$.*

Proof. We consider two cases.

Case 1. $\nu \geqslant \lambda$. The proof proceeds by transfinite induction on λ, having observed that if $\lambda = \lambda^{\#}$, then the result holds trivially.

Suppose that the result holds for all β such that $\beta < \lambda$.

If $\varkappa < \lambda$, then ν is characterized at $(\varkappa, \varkappa)$ and so, by the induction hypothesis, ν is characterized at $(\varkappa, \varkappa^{\#})$, and hence at $(\varkappa, \lambda^{\#})$ – since $\varkappa < \lambda$ implies $\varkappa^{\#} \leqslant \lambda^{\#}$. Thus, we may suppose that $\varkappa \geqslant \lambda$.

(a) Suppose that $\lambda = \beta^+$ for some β. Since $\varkappa \geqslant \beta^+$, β^+ is characterized at $(\varkappa, \beta)$, and so, by the induction hypothesis, at $(\varkappa, \beta^{\#})$. Using Corollary 2.1 and the fact that $(\beta^+)^{\#} = \beta^{\#}$, if ν is characterized at $(\varkappa, \beta^+)$, then ν is characterized at $(\varkappa, (\beta^+)^{\#})$.

(b) Suppose that $\lambda > \lambda^{\#}$, $\lambda > \mathrm{cf}(\lambda)$, and

$$\lambda = \sum_{\xi < \mathrm{cf}(\lambda)} a_\xi$$

for some set of cardinals $\{a_\xi : \xi < \mathrm{cf}(\lambda)\}$ such that $a_\xi < \lambda$ for all $\xi < \mathrm{cf}(\lambda)$. For $\xi < \mathrm{cf}(\lambda)$, a_ξ is characterized at (a_ξ, a_ξ), and hence at $(\varkappa, a_\xi)$ since, by assumption, $\varkappa \geqslant \lambda > a_\xi$. It follows from the induction hypothesis that a_ξ is characterized at $(\varkappa, a_\xi^{\#})$, and hence at $(\varkappa, \lambda^{\#})$ – again, since $a_\xi \leqslant \lambda$ implies $a_\xi^{\#} \leqslant \lambda^{\#}$. Similarly, $\mathrm{cf}(\lambda)$ is characterized at $(\varkappa, \lambda^{\#})$.

It now follows, using a method from [4], p. 267, that λ can be characterized by a pair (T, Σ), where

$$|T| \leqslant \varkappa\mathrm{cf}(\lambda) + \mathrm{cf}(\lambda) = \varkappa \quad \text{and} \quad |\Sigma| \leqslant \lambda^{\#}\mathrm{cf}(\lambda) + \mathrm{cf}(\lambda) = \lambda^{\#} + \mathrm{cf}(\lambda).$$

That is to say, λ can be characterized at $(\varkappa, \lambda^{\#}+\mathrm{cf}(\lambda))$. Now $\lambda^{\#}+\mathrm{cf}(\lambda) < \lambda$ using the assumptions on λ; so, by the induction hypothesis, λ can be characterized at $(\varkappa, \lambda^{\#})$.

Suppose that ν is characterized at $(\varkappa, \lambda)$. By the above, λ can be characterized at $(\varkappa, \lambda^{\#})$, and so ν can be characterized at $(\varkappa, \lambda^{\#})$ by Corollary 2.1.

Case 2. $\nu < \lambda$. Clearly, ν can be characterized at (ν, ν) and so, by Case 1, at $(\nu, \nu^{\#})$, and hence at $(\varkappa, \lambda^{\#})$ if $\nu \leqslant \varkappa$. If $\nu > \varkappa$, then ν can be characterized at $(\varkappa, \nu)$, and so at $(\varkappa, \nu^{\#})$ and $(\varkappa, \lambda^{\#})$.

This concludes the proof of the theorem.

COROLLARY 2.2. *For cardinals $\varkappa$ and λ, $\mu_1(\varkappa, \lambda) = \mu_1(\varkappa, (\lambda^{\#})^{+})$ if $\lambda \neq \lambda^{\#}$.*

This follows immediately from the definition of μ_1 and Theorem 2.2.

The above result can be improved. To do this, we first observe the following

LEMMA 2.1. *If ν is characterized at $(\varkappa, \lambda)$ and $\lambda \leqslant 2^{\alpha} \leqslant \nu$ for some cardinal α, then ν is characterized at $(\varkappa, \alpha)$.*

Proof. Obviously, we may assume that $\alpha < \lambda$. Now, 2^{α} can be characterized at (α, α) (see [4], p. 265-273), so if $\alpha \leqslant \varkappa$, then 2^{α} can be characterized at $(\varkappa, \alpha)$ and the result follows by applying Corollary 2.1.

In the remaining case, namely $\alpha > \varkappa$, the result is trivial.

We now state the main theorem of this section.

THEOREM 2.3. *For cardinals $\varkappa$ and λ, if $\lambda \neq \lambda^{\#\#}$, then*

$$\mu_1(\varkappa, \lambda) = \mu_1(\varkappa, (\lambda^{\#\#})^{+}).$$

Furthermore, if $\varkappa$ is smaller than or equal to λ, then

$$\mu_1(\varkappa, \lambda) = \mu_1(\varkappa, \varkappa).$$

Proof. The second result is obvious.

To obtain the first result, the induction in the proof of Theorem 2.2 is extended using Lemma 2.1. So, a further case, 3, is obtained:

Case 3. Suppose that $\lambda = 2^{\beta}$ for some cardinal $\beta < \lambda$, and ν is characterized at $(\varkappa, \lambda)$. By Lemma 2.1, ν is characterized at $(\varkappa, \beta)$.

So, in particular, we note that if θ_0 is the first strongly inaccessible cardinal and $\aleph_0 < \lambda < \theta_0$, then $\mu_1(\varkappa, \lambda) = \mu_1(\varkappa, \aleph_1)$.

Similarly, if no strongly inaccessible cardinal exists, then $\mu_1(\varkappa, \lambda) = \mu_1(\varkappa, \aleph_1)$ for all $\lambda > \aleph_0$.

3. Omitting sets of types.

We begin by extending Definition 1.1.

Definition 3.1. Let $\mathfrak{S}$ denote a set of types. Then the cardinal λ is *characterized by the pair* $(T, \mathfrak{S})$ if every model of T, which omits every type in $\mathfrak{S}$, has power smaller than λ and, for every cardinal ν smaller than λ, T has a model of power at least ν which omits every type in $\mathfrak{S}$.

Definition 3.2. For a cardinal θ, θ is *characterized at* $(\varkappa, \lambda, \nu)$ (by the pair $(T, \mathfrak{S})$) if $\mathfrak{S}$ is a set of types such that $|\mathfrak{S}|$ is at most equal to λ, $T \cup \bigcup \mathfrak{S}$ is contained in a language of power at most $\varkappa$, each member of $\mathfrak{S}$ has power at most ν, and $(T, \mathfrak{S})$ characterizes θ.

Definition 3.3. For cardinals $\varkappa$, λ, and ν, we define

$$\mu_2(\varkappa, \lambda, \nu) = \sup\{\theta : \theta \text{ is characterized at } (\varkappa_1, \lambda_1, \nu_1) \text{ for some } \varkappa_1 < \varkappa, \lambda_1 < \lambda \text{ and } \nu_1 < \nu\}.$$

In this section we examine $\mu_2(\varkappa, \lambda, \nu)$ — the *second refined Morley number*. A proof of the following theorem may be found in [2] and [7].

THEOREM 3.1. $\mu_2(\varkappa^+, (2^\varkappa)^+, \varkappa^+) = 2(\aleph_0, (2^\varkappa)^+)$.

Next, observe that μ_2 is a non-decreasing function with respect to each variable. Put

$$\sum_{\lambda<\varkappa} \nu^\lambda = \nu^{<\varkappa}.$$

THEOREM 3.2. *If $\varkappa$ is a limit cardinal, then*

(i) $$\mu_2(\varkappa, 2^{<\varkappa}, \varkappa) = 2\Big(\aleph_0, \sum_{\lambda<\varkappa}(2^\lambda)^+\Big) \quad \textit{if } 2^\lambda < 2^{<\varkappa} \textit{ for all } \lambda < \varkappa$$

and

(ii) $$\mu_2\big(\varkappa, (2^{<\varkappa})^+, \varkappa\big) = 2(\aleph_0, 2^{<\varkappa}) \quad \textit{if } 2^\lambda = 2^{<\varkappa} \textit{ for some } \lambda < \varkappa.$$

Proof. (i) First,

$$\begin{aligned}\mu_2(\varkappa, 2^{<\varkappa}, \varkappa) &= \sup\{\theta : \theta \text{ is characterized at } (\varkappa_1, \lambda_1, \nu_1) \text{ for some } \varkappa_1 < \varkappa, \lambda_1 < 2^{<\varkappa} \text{ and } \nu_1 < \varkappa\} \\ &= \sup\{\mu_2(\lambda^+, (2^\lambda)^+, \lambda^+) : \lambda < \varkappa\} \quad \text{if } 2^\lambda < 2^{<\varkappa} \text{ for all } \lambda < \varkappa.\end{aligned}$$

The result follows, after an application of Theorem 3.1, from the fact that

$$2\Big(\aleph_0, \sum_{\lambda<\varkappa}(2^\lambda)^+\Big) = \sup\{2(\aleph_0, (2^\lambda)^+) : \lambda < \varkappa\}.$$

To see this, observe that since $2^\lambda < 2^{<\varkappa}$ for all $\lambda < \varkappa$, there is a sequence $\langle \lambda_\xi : \xi < \mathrm{cf}(\varkappa)\rangle$ which is cofinal with $\varkappa$ and such that $2^{\lambda_\xi} < 2^{\lambda_\zeta}$ for $\xi < \zeta < \mathrm{cf}(\varkappa)$. Then

$$\sup\{2(\aleph_0, (2^\lambda)^+) : \lambda < \varkappa\} = 2\Big(\aleph_0, \sum_{\lambda<\varkappa}(2^\lambda)^+\Big)$$

from the definition of $2(\alpha, \beta)$, since $\varkappa$ is a limit cardinal.

(ii) If $2^{\lambda} = 2^{<\varkappa}$ for some $\lambda < \varkappa$, then

$$\mu_2\big(\varkappa, (2^{<\varkappa})^+, \varkappa\big) = \sup\{\mu_2\big(\lambda^+, (2^{\lambda})^+, \lambda^+\big) : \lambda < \varkappa\}$$
$$= \sup\{2\big(\aleph_0, (2^{\lambda})^+\big) : \lambda < \varkappa\} = 2\big(\aleph_0, (2^{<\varkappa})^+\big).$$

THEOREM 3.3. (i) $\mu_2(\varkappa^+, \lambda, \nu) = \mu_2\big(\varkappa^+, (\varkappa^{<\nu})^+, \nu\big)$ *if* $\lambda > \varkappa^{<\nu}$.

(ii) $\mu_2(\varkappa, \lambda, \nu) = \mu_2(\varkappa, \lambda, \varkappa)$ *if* $\nu \geqslant \varkappa$.

These follow directly from the definition of $\mu_2(\varkappa, \lambda, \nu)$.

THEOREM 3.4. *If $\varkappa$ is a limit cardinal and*

$$\lambda > \sum_{a<\varkappa} a^{<\nu},$$

then

$$\mu_2(\varkappa, \lambda, \nu) = \mu_2\Big(\varkappa, \sum_{a<\varkappa} (a^{<\nu})^+, \nu\Big).$$

Proof. Since μ_2 is non-decreasing, we have

$$\mu_2(\varkappa, \lambda, \nu) \geqslant \mu_2\Big(\varkappa, \sum_{a<\varkappa} (a^{<\nu})^+, \nu\Big)$$

by the assumption on $\varkappa$, λ, and ν. Conversely,

$$\begin{aligned}\mu_2(\varkappa, \lambda, \nu) &= \sup\{\mu_2(a^+, \lambda, \nu) : a < \varkappa\}\\ &= \sup\{\mu_2\big(a, (a^{<\nu})^+, \nu\big) : a < \varkappa\} \quad \text{by Theorem 3.3 (i)}\\ &\leqslant \sup\{\mu_2\big(\varkappa, (a^{<\nu})^+, \nu\big) : a < \varkappa\}\\ &= \mu_2\Big(\varkappa, \sum_{a<\varkappa} (a^{<\varkappa})^+, \nu\Big).\end{aligned}$$

THEOREM 3.5. (i) *If $\lambda \leqslant \nu \leqslant \varkappa$, then*

$$\mu_2(\varkappa, \lambda, \nu) = \mu_1(\varkappa, \nu).$$

(ii) *If $\nu < \lambda \leqslant \varkappa$, then*

$$\mu_2(\varkappa, \lambda, \nu) = \mu_1(\varkappa, \lambda).$$

Proof. Both results follow from the fact that a set of a types each of power less than β may be replaced by a single type of power less than or equal to $\max\{a, \beta\}$ — the technique for doing this may be found in [4], p. 267.

From the results above, μ_2 can be evaluated (considered as a function on the class of cardinals) if μ_1 is known and also

(1) $\mu_2(\varkappa^+, \lambda, \nu)$ is known for all $\varkappa$, λ, and ν such that

$$\varkappa < \lambda \leqslant (2^{\varkappa})^+, \quad \nu \leqslant \varkappa^+, \quad \text{and} \quad \varkappa^{<\nu} > \varkappa;$$

(2) $\mu_2(\varkappa, \lambda, \nu)$ is known for $\varkappa$ a limit cardinal and all λ and ν such that

$$\varkappa \leqslant \lambda \leqslant 2^{<\varkappa}, \quad \nu \leqslant \varkappa, \quad \text{and} \quad \theta^{<\nu} > \varkappa \text{ for some } \theta < \varkappa.$$

If the generalized continuum hypothesis is assumed, then the results above simplify considerably since $2^a = a^+$ and $2^{<a} = a$ for all cardinals a. In particular, μ_2 can then be evaluated entirely if μ_1 is known.

The author would like to thank the referee for his many helpful comments and, in particular, for the simplified proof of Theorem 3.4.

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Reçu par la Rédaction le 17. 4. 1974;
en version définitive le 20. 7. 1976