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INDEPENDENT SUBALGEBRAS OF A GENERAL ALGEBRA

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1. In this note we use the terminology and notation of [4], [5] and [1], but we say "general algebra" instead of "abstract algebra". Some of the theorems obtained here arose from questions posed to us by Professor Edward Marczewski. He proposed the following definition:

Two subalgebras B_1 and B_2 of a general algebra $\mathfrak{A} = (A; \boldsymbol{F})$ are *independent* if, for every pair of homomorphisms $h_i: B_i \to A$ $(i = 1, 2)$, there exists a homomorphism $h: C(B_1 \cup B_2) \to A$ such that $h|_{B_i} = h_i$ $(i = 1, 2)$.

2. We extend Marczewski's definition for a set of subalgebras of $\mathfrak{A}$.

Definition. A set $\mathscr{I}$ of subalgebras of an algebra $\mathfrak{A} = (A; \boldsymbol{F})$ is *independent* if, for every family of homomorphisms $h_B: B \to A$, $B \in \mathscr{I}$, there exists a homomorphism

$$h: C\Big(\bigcup_{B \in \mathscr{I}} B\Big) \to A$$

such that $h|_B = h_B$ for $B \in \mathscr{I}$.

We denote by $2^{\mathfrak{A}}$ the set of all subalgebras of $\mathfrak{A}$ and we use the notation $\mathscr{I} \in \mathbf{ind}\, 2^{\mathfrak{A}}$ (or, shortly, $\mathscr{I} \in \mathbf{ind}$) if $\mathscr{I}$ is an independent set of subalgebras of $\mathfrak{A}$.

Example 1. The notion of independence of subalgebras is connected with the notion of a free product of algebras (see [3] and [7]) similarily as the independence of elements is connected with the notion of a free generated algebra. Recall that an algebra A is a *$\mathscr{K}$-free product* of its subalgebras B and C if $A = C(B \cup C)$ and, for every pair of homomorphisms $h_B: B \to D$ and $h_C: C \to D$, there is a homomorphism h: $A \to D$ such that $h|_B = h_B$ and $h|_C = h_C$, where D is an arbitrary algebra from the class $\mathscr{K}$ of algebras of the same similarity type as the algebra A. This definition makes sense also for algebras without determined similarity type (non-indexed algebras) whenever A, B and C are subalgebras of a certain algebra $\mathfrak{A}_0 = (A_0; F)$ and $\mathcal{K} \subset 2^{\mathfrak{A}_0}$. One can speak about a $\mathcal{K}$-free product of a set of subalgebras of $\mathfrak{A}_0$.

Note that *the set $\mathcal{I}$ of subalgebras of $\mathfrak{A}_0$ is independent if and only if the subalgebra*

$$A = C\Big(\bigcup_{B\in\mathcal{I}} B\Big)$$

is a $\mathcal{K}$-free product of subalgebras $B\in\mathcal{I}$ for any $\mathcal{K}$ such that $A_0\in\mathcal{K}\subset 2^{\mathfrak{A}_0}$. In particular, it is true for $\mathcal{K} = 2^{\mathfrak{A}_0}$ and for $\mathcal{K} = \{A_0\}$ $(=\{\mathfrak{A}_0\})$.

Example 2. The following notion of independence (which we shall call *$\mathcal{B}$-independence*) of subalgebras of a Boolean algebra is well known:

The indexed set $\{B_t\}_{t\in T}$ of subalgebras of a Boolean algebra

$$\mathfrak{A} = (A; \{\cup, \cap, ', 0, 1\})$$

is *$\mathcal{B}$-independent* if

$$\bigcap_{t\in T_0} b_t \neq 0$$

for an arbitrary finite $T_0 \subset T$ and $b_t \neq 0$, $b_t \in B_t$ (see [6], Section 13).

Theorem 13.1 of [6] shows that any $\mathcal{B}$-independent set of subalgebras is independent in our sense. Note that the converse is also true. Indeed, let $\mathcal{I}\in \mathbf{ind}\, 2^{\mathfrak{A}}$, where $\mathfrak{A}$ is a Boolean algebra, and suppose that $\mathcal{I} = \{B: B\in\mathcal{I}\}$ is not $\mathcal{B}$-independent. Hence there are elements $b_1, \ldots, b_n$ such that $0 \neq b_i \in B_i \in \mathcal{I}$, $B_i \neq B_j$ for $i \neq j$, and $b_1 \cap \ldots \cap b_n = 0$. As it is known, for any b_i there exists an ultrafilter F_i in B_i such that $b_i \in F_i$ for $i = 1, \ldots, n$. Define homomorphisms $h_i: B_i \to A$ by

$$h_i(a) = \begin{cases} 1 & \text{for } a\in F_i, \\ 0 & \text{otherwise} \end{cases}$$

(h_i is a homomorphism, for F_i is an ultrafilter). Note that $h_i(b_i) = 1$, and thus there exists no homomorphism

$$h: C\Big(\bigcup_{B\in\mathcal{I}} B\Big) \to A$$

such that $h_i = h|_{B_i}$ for $i = 1, \ldots, n$, because otherwise

$$1 = h_1(b_1) \cap \ldots \cap h_n(b_n) = h(b_1 \cap \ldots \cap b_n) = h(0) = 0.$$

Example 3. If $\mathfrak{A}$ is a unary algebra, then, as it is easy to see, any set of pairwise disjoint subalgebras of $\mathfrak{A}$ is independent. The following example shows that, in general, this is not true for non-unary algebras. Viz. $A = \{a, b, c\}$ and $\circ$ is a binary operation such that $x\circ c = c\circ x = c$ for every $x\in A$ and $x\circ y = x$ for $x \neq c$ and $y \neq c$. Let

$$h_a: \{a\} \to \{c\} \quad \text{and} \quad h_b: \{b\} \to \{b\}.$$

If there were a homomorphism $h\colon \{a, b\} \to A$ such that $h(a) = c$ and $h(b) = b$, then

$$b = h(b) = h(b \circ a) = h(b) \circ h(a) = h(b) \circ c = c,$$

which is a contradiction.

3. It follows immediately from the definition of the independence of subalgebras $\{B\colon B \in \mathscr{I}\}$ that, for every family of homomorphisms $\{h_B\}$, the homomorphism

$$(+) \qquad h\colon C\Big(\bigcup_{B \in \mathscr{I}} B\Big) \to A$$

such that $h|_B = h_B$ is unique. It is also obvious that the independence of a set $\mathscr{I}$ of subalgebras is a hereditary property with respect to subsets of $\mathscr{I}$. It can be easily shown that the independence of a set $\mathscr{I}$ of subalgebras of $\mathfrak{A}$ is an invariant property with respect to meromorphisms (injective homomorphisms). Namely, *if* $\mathfrak{A} = (A; F)$, $\mathscr{I} \subset 2^{\mathfrak{A}}$ *and mapping* $(+)$ *is a meromorphism, then* $\mathscr{I} \in \mathbf{ind}\, 2^{\mathfrak{A}}$ *if and only if* $\{h(B)\colon B \in \mathscr{I}\} \in \mathbf{ind}\, 2^{\mathfrak{A}}$.

The following theorem implies that the independence of a set of subalgebras (like the independence of a set of elements; see [4]) is a property of finitary character:

THEOREM 1. *Let* $\mathfrak{A} = (A; F)$ *and* $\mathscr{I} \subset 2^{\mathfrak{A}}$. *Then* $\mathscr{I} \in \mathbf{ind}\, 2^{\mathfrak{A}}$ *if and only if*

(*) *for any family of homomorphisms* $h_B\colon B \to A$, $B \in \mathscr{I}$, *if* f *and* g *are* n*-ary and* m*-ary algebraic operations in* $\mathfrak{A}$, *and* $a_i \in A_i \in \mathscr{I}$, $b_j \in B_j \in \mathscr{I}$ $(i = 1, \ldots, n;\ j = 1, \ldots, m;\ n, m = 1, 2, \ldots)$, *then the equality*

$$f(a_1, \ldots, a_n) = g(b_1, \ldots, b_m)$$

implies

$$f\big(h_{A_1}(a_1), \ldots, h_{A_n}(a_n)\big) = g\big(h_{B_1}(b_1), \ldots, h_{B_m}(b_m)\big).$$

Proof. The necessity of (*) is obvious; we shall show that (*) is sufficient. We define mapping $(+)$. If

$$a \in C\Big(\bigcup_{B \in \mathscr{I}} B\Big),$$

then there exist an algebraic operation f and elements $a_i \in A_i \in \mathscr{I}$, $i = 1, \ldots, n$, such that $a = f(a_1, \ldots, a_n)$. Put

$$h(a) = f\big(h_{A_1}(a_1), \ldots, h_{A_n}(a_n)\big).$$

Condition (*) guarantees that the definition of h is correct. It is easy to see that h is a homomorphism and $h|_B = h_B$ for $B \in \mathscr{I}$, which completes the proof.

COROLLARY 1. *The independence of a set of subalgebras is a property of finitary character (i.e.,* $\mathscr{I} \in \mathbf{ind}$ *if and only if* $\mathscr{I}_0 \in \mathbf{ind}$ *for any finite* $\mathscr{I}_0 \subset \mathscr{I}$*).*

Indeed, this corollary follows from (*).

COROLLARY 2. *Every independent set of subalgebras is contained in some maximal independent set of subalgebras.*

In fact, it follows from Corollary 1 and Tukey's Lemma.

COROLLARY 3. *If* $\mathscr{I} \in \mathbf{ind}\, 2^{\mathfrak{A}}$, $|\mathscr{I}| \geqslant 2$, *and* $a \in B$ *for every* $B \in \mathscr{I}$, *then* $h_B(a) = a$ *for every homomorphism* $h_B\colon B \to A$, $B \in \mathscr{I}$.

Indeed, let $B, B_1 \in \mathscr{I}$ and let h_{B_1} be the identity homomorphism. Putting for f and g in $(*)$ the unary trivial operation we obtain $a = h_{B_1}(a) = h_B(a)$.

We call an element $a \in A$ a *quasi-constant with respect to subalgebra* B *of* A (see [2]) if $a \in B$ and $h(a) = a$ for every homomorphism $h\colon B \to A$. Using this notion, we can formulate Corollary 3 in the following way:

If an element belongs to the intersection of a family of independent subalgebras, then it is a quasi-constant with respect to each of them.

In particular, in algebras with one constant c (e.g., in groups, rings and modules) this intersection is always equal to $\{c\}$. In algebras with a basis it is equal to $C(\emptyset)$, and in idempotent algebras it is the empty set (see [2]).

4. Now we examine some connections and analogies between the independence of a set of subalgebras and the independence of a set of elements.

THEOREM 2. *If* $\mathfrak{A} = (A; F)$ *and* I *is an independent subset of* A, *then* $\{C(\{a\})\colon a \in I\} \in \mathbf{ind}$.

Proof. If $h_a\colon C(\{a\}) \to A$, $a \in I$, is a family of homomorphisms, then we define the mapping $p\colon I \to A$ by

$$p(a) = h_a(a) \quad \text{for every } a \in I.$$

Since I is independent, this mapping can be extended to a homomorphism $h\colon C(I) \to A$. Note that $h|_{C(\{a\})} = h_a$, which completes the proof.

The following example shows that the converse is not true:

Let $\mathfrak{A} = (A; \{f\})$ be a unary algebra, where $A = \{a, b, c\}$, $f(a) = b$, $f(b) = a$ and $f(c) = c$. $\{\{a, b\}, \{c\}\}$ is an independent set of subalgebras (it follows from the remark in Example 3). However, $\{a, c\}$ is not an independent set of elements, because c is a self-dependent element.

Nevertheless, we have

THEOREM 3. *Let* I *be a set of non-self-dependent elements of an algebra* $\mathfrak{A}$. *If* $\{C(\{a\})\colon a \in I\} \in \mathbf{ind}$, *then* I *is independent.*

Proof. Let $p\colon I \to A$. Since every element $a \in I$ is non-self-dependent, $p|_{\{a\}}$ can be extended to a homomorphism $h_a\colon C(\{a\}) \to A$ for every $a \in I$. And since $\{C(\{a\})\colon a \in I\} \in \mathbf{ind}$, there exists a common extension of h_a $(a \in I)$ to a homomorphism $h\colon C(I) \to A$. Obviously, $h|_I = p$, which completes the proof.

Let H_* be the smallest family of mappings from subsets of algebra $\mathfrak{A} = (A; F)$ into A, containing all mappings, which can be extended to homomorphisms and closed with respect to restrictions and "stickings" of mappings on disjoint subsets (see [1], p. 28). The last two theorems can be formulated more briefly using the following notion of H_*-*independence* [1]:

$$\{C(\{a\})\colon a \in I\} \in \mathbf{ind}\, 2^{\mathfrak{A}} \quad \textit{if and only if} \quad I \in \mathbf{Ind}(\mathfrak{A}, H_*).$$

The following more general theorem can be obtained in the similar way as Theorems 2 and 3:

THEOREM 4. *Let* $\{I_t\}_{t \in T}$ *be a family of independent and pairwise disjoint subsets of an algebra* $\mathfrak{A}$. *Then*

$$\{C(I_t)\colon t \in T\} \in \mathbf{ind}\, 2^{\mathfrak{A}}$$

if and only if $\bigcup_{t \in T} I_t$ *is an independent set in* $\mathfrak{A}$.

The next theorem is analogous to the "exchange theorem" of [4] (see (ii), p. 58).

THEOREM 5. *Let* $(\mathscr{I}_0 \cup \mathscr{I}_1) \in \mathbf{ind}$, $\mathscr{I}_0 \cap \mathscr{I}_1 = \emptyset$, $\mathscr{I}_2 \in \mathbf{ind}$ *and let a set* $\mathscr{I}_1^0 \subset \mathscr{I}_1$ *exist such that*

$$C\Big(\bigcup_{B \in \mathscr{I}_1^0} B\Big) = C\Big(\bigcup_{B \in \mathscr{I}_2} B\Big).$$

Then $(\mathscr{I}_0 \cup \mathscr{I}_2) \in \mathbf{ind}$.

Proof. For $D \in (\mathscr{I}_0 \cup \mathscr{I}_2)$, let $h_D\colon D \to A$ be a family of homomorphisms. Since $\mathscr{I}_2 \in \mathbf{ind}$, there exists a homomorphism

$$h\colon C\Big(\bigcup_{D \in \mathscr{I}_2} D\Big) \to A$$

such that $h|_D = h_D$ for any $D \in \mathscr{I}_2$. Obviously, h is also a homomorphism from $C\big(\bigcup_{B \in \mathscr{I}_1^0} B\big)$ into A. Put $h_B^0 = h|_B$ for $B \in \mathscr{I}_1^0$. Since $\mathscr{I}_0 \cap \mathscr{I}_1^0 = \emptyset$ and $(\mathscr{I}_0 \cup \mathscr{I}_1^0) \in \mathbf{ind}$, there exists a homomorphism

$$h^0\colon C\Big(\bigcup_{B \in (\mathscr{I}_0 \cup \mathscr{I}_1^0)} B\Big) \to A$$

such that $h^0|_B = h_B^0$ for $B \in \mathscr{I}_1^0$ and $h^0|_D = h_D$ for $D \in \mathscr{I}_0$. We show that $h^0|_D = h_D$ for $D \in (\mathscr{I}_0 \cup \mathscr{I}_2)$. It is obvious for $D \in \mathscr{I}_0$. If $D \in \mathscr{I}_2$, then

$$D \subset C\Big(\bigcup_{B \in \mathscr{I}_1^0} B\Big),$$

and thus for $d \in D$ there exist an n-ary algebraic operation f and elements $b_i \in B_i \in \mathscr{I}_1^0$ $(i = 1, \ldots, n)$ such that $d = f(b_1, \ldots, b_n)$. Thus we have

$$h^0(d) = h^0\big(f(b_1, \ldots, b_n)\big) = f\big(h^0(b_1), \ldots, h^0(b_n)\big) = f\big(h(b_1), \ldots, h(b_n)\big)$$
$$= h\big(f(b_1, \ldots, b_n)\big) = h(d) = h_D(d).$$

This completes the proof of Theorem 5.

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