COLLOQUIUM MATHEMATICUM

VOL. XLVIII 1984 FASC. 2

# GENERALIZATION OF WEYL–VON NEUMANN–BERG THEOREM FOR THE CASE OF NORMAL OPERATOR-VALUED HOLOMORPHIC FUNCTIONS

BY

BOGDAN BARAN (KRAKÓW)

The theorem of Weyl–von Neumann states that each hermitian operator in a separable Hilbert space can be written as a sum of two operators: diagonal and compact. D. Berg (1971) and P. R. Halmos (1972) proved this property in the case of normal operator.

In this note we intend to establish this fact for any normal operator-valued holomorphic function $f$ (in a separable Hilbert space). We shall prove that there exist two holomorphic functions $g$, $h$ such that $f = g+h$, where $g$ is diagonal operator-valued and $h$ is compact operator-valued. The proof is based on the Globevnik–Vidav lemma [3] and on Brown–Douglas–Fillmore lemmas [1].

**1.** Let $H$ be a complex Hilbert space. By $\mathcal{N}$ we denote the set of all normal operators in $H$. First we give the following lemma (due to Globevnik–Vidav [3] for $n = 1$; however, it is easy to see that it holds also for $n > 1$):

LEMMA 1. *Let $D \subset C^n$ be the unit polydisc and let $f\colon D \to L(H)$ be a holomorphic function such that $f(D) \subset \mathcal{N}$ and*

$$f(\gamma) = \sum_{\alpha\in N^n} A_\alpha \gamma^\alpha \quad \textit{for all } \gamma\in D, \tag{1}$$

*where $A_\alpha \in L(H)$. Then for every $\alpha$, $\beta\in N^n$ we have*

$$A_\alpha A_\beta = A_\beta A_\alpha \quad \textit{and} \quad A_\alpha A_\beta^* = A_\beta^* A_\alpha.$$

As a corollary we obtain

LEMMA 2. *Let $\Omega$ be an open and connected subset of $C^n$ and let $f\colon \Omega \to L(H)$ be a holomorphic function in $\Omega$. Assume that there exist $\gamma_0 \in \Omega$ and a neighbourhood $U_{\gamma_0}$ of $\gamma_0$ contained in $\Omega$ such that $f(U_{\gamma_0}) \subset \mathcal{N}$. Then $f(\Omega) \subset \mathcal{N}$ and for any $\gamma$, $\eta\in\Omega$ we have $f(\gamma)f(\eta) = f(\eta)f(\gamma)$.*

Proof. We may assume that $\gamma_0 = 0$ and $U_{\gamma_0} = D$, $D$ being the unit polydisc in $C^n$. Then (1) holds. Moreover, $\{A_\alpha\}_{\alpha\in N^n}$ is a normal subset of $L(H)$ ($A \subset L(H)$ is called *normal* if $A \cup A^*$ is commutative).

Now, consider an algebra $\mathscr{A}$ generated in $(L(H), \|\cdot\|)$ by $\{A_\alpha\}_{N^n} \cup \cup \{A_\alpha^*\}_{N^n} \cup \{I\}$. It is a commutative $C^*$-algebra. Obviously, $f(D) \subset \mathscr{A}$. The algebra $\mathscr{A}$, as a vector space, is a closed linear subspace of $L(H)$. Therefore, there exists a set $T$ of linear continuous functionals on $L(H)$ such that

$$(2) \qquad A \in \mathscr{A} \Leftrightarrow \forall u \in T \ u(A) = 0.$$

Since $f(D) \subset \mathscr{A}$, we have $u \circ f(\gamma) = 0$ for every $\gamma \in D$. The function $u \circ f$ is holomorphic in $\Omega$ (connected), so from the identity principle we have $u \circ f \equiv 0$ in $\Omega$. In virtue of (2), $f(\Omega) \subset \mathscr{A}$. Since $\mathscr{A}$ is contained in $\mathscr{N}$, $f(\Omega) \subset \mathscr{N}$, which completes the proof.

The following lemma has been proved by Brown et al. [1]:

LEMMA 3. *Let $\mathscr{A}$ be a commutative, countably generated $C^*$-algebra. Then there exists a commutative $C^*$-algebra $Z$ such that $\mathscr{A} \subset Z$ and $Z$ is generated by one element ($\mathscr{A}$ and $Z$ are contained in certain $L(H)$).*

Proof. We denote by $X$ the spectrum of $\mathscr{A}$. From our assumptions it follows that $X$ is a compact metric space, so $X$ is separable. Let $\{U_n\}_{n\in N}$ be a base of topology in $X$. Let $T\colon C(X) \to \mathscr{A}$ be a $*$-isometrical isomorphism (by the spectral theorem). We have

$$Tf = \int_X f(\lambda) E(d\lambda) \qquad \text{for all } f \in C(X),$$

where $E$ is a spectral measure on $X$. We define $Z$ as a closed algebra generated by $\{E_n\}_N \cup \{I\}$, where $E_n = E(U_n)$. The algebra $Z$ is generated by the hermitian operator

$$W = \sum_{n=0}^{\infty} 3^{-n}(2E_n - I).$$

One can show that $\mathscr{A} \subset Z$ and the proof is complete.

The next lemma depends on the Weyl theorem concerning the Weyl spectrum. Let $A \in L(H)$. Let

$$\omega(A) = \bigcap_{K\in\mathscr{K}} \sigma(A+K), \qquad \text{where } \mathscr{K} = \{K \in L(H)\colon K \text{ compact}\}.$$

The Weyl theorem states that if $A$ is normal (even hyponormal – see [2]), then $\omega(A)$ consists of all points of $\sigma(A)$ except for isolated eigenvalues of finite multiplicity.

LEMMA 4. *Let $Z$ be a commutative $C^*$-algebra generated by one hermitian element $W$. Assume that $Z \subset L(H)$, where $H$ is a separable Hilbert space. Then each operator $S \in Z$ can be written as a sum $D_s + K_s$, where $D_s$ is diagonal and $K_s$ compact.*

Proof (see [1]). Let $Y$ be the spectrum of $Z$. By the Weyl–von Neumann theorem there exist operators $D$ and $K$, $D$ diagonal and $K$ compact, such that $W = D+K$. Let $U$ be the spectrum of $D$. Putting $A = D$ in the Weyl theorem we infer that $Y$ contains $U$ except for isolated eigenvalues of finite multiplicity of $D$. This allows us to construct a mapping $\beta$ from $C(Y)$ into $C(Y \cup U)$ which is a contractive $*$-homomorphism. Consider now some $f \in C(Y)$. We write $f^0 = \beta(f) \in C(Y \cup U)$. By the Weierstrass theorem we obtain a sequence of polynomials of two real variables $\{p_n\}_N$ such that $p_n \underset{Y \cup U}{\rightrightarrows} f^0$. Hence

$$p_n \underset{U}{\rightrightarrows} f^0|_U,$$

and then $p_n(D) \to f^0|_U(D)$. Since $p_n(D)$ is diagonal, so is $f^0|_U(D)$.

We also have $p_n(W) \to f(W)$, so $p_n(W) - p_n(D)$ is compact and convergent to some compact operator $K_f$. It follows that $f(W) = f^0|_U(D) + K_f$. Now we may put $f^0|_U(D) = D_s$ and $K_f = K_s$ for $S = f(W)$, and the proof is complete.

2. In virtue of the above considerations we can formulate our main result.

THEOREM. *If $H$ is a separable Hilbert space, $\Omega \subset C^n$ is an open set, and $F\colon \Omega \to \mathcal{N} \subset L(H)$ is a holomorphic function, then there exist holomorphic mappings $g$, $h\colon \Omega \to L(H)$ such that, for any $z \in \Omega$, $g(z)$ is diagonal, $h(z)$ is compact, and $F(z) = g(z) + h(z)$.*

Proof. (a) Let $\Omega$ be connected.

We choose an arbitrary point $\gamma_0 \in \Omega$ and a neighbourhood $U_{\gamma_0}$ of $\gamma_0$ in $\Omega$ such that

$$F(z) = \sum_{\alpha \in N^n} A_\alpha (z-\gamma_0)^\alpha$$

whenever $z \in U_{\gamma_0}$. By Lemma 1 the set $\{A_\alpha\}_{\alpha \in N^n}$ is normal and generates a commutative $C^*$-algebra. We have $F(U_{\gamma_0}) \subset \mathcal{A}$, so $F(\Omega) \subset \mathcal{A}$. By Lemma 3 there exists a $C^*$-algebra $Z$ which is commutative and contains $\mathcal{A}$. Moreover, $Z$ is generated by a hermitian operator $W$. By Lemma 4, for any element of $Z$ the assertion of Weyl–von Neumann theorem holds true. Let $X$ and $Y$ be the spectra of $\mathcal{A}$ and $Z$, respectively, and let $T\colon C(X) \to \mathcal{A}$ and $Q\colon C(Y) \to Z$ be the corresponding $*$-isometrical isomorphisms. Put $\alpha = Q^{-1} T$. Let $\beta\colon C(Y) \to C(Y \cup U)$ be as in the proof of Lemma 4, where $U$ is the spectrum of $D$ (here $W = D+K$). Let $f \in C(X)$. We put $\tilde{f} = (\beta \circ \alpha(f))|_U$. For any $z \in \Omega$ we have $F(z) = \tilde{f}_z(D) + K_z$, where $f_z = T^{-1}(F(z))$. The mapping $\Omega \ni z \mapsto \tilde{f}_z(D)$ is holomorphic. Indeed, for $z_0 \in \Omega$, $z \in \Omega$ we have (for simplicity we put $n = 1$)

$$\frac{\tilde{f}_z(D)-\tilde{f}_{z_0}(D)}{z-z_0} = \frac{\big(\beta\circ\alpha(T^{-1}F(z))\big)\big|_U(D)-\big(\beta\circ\alpha\big(T^{-1}(F(z_0))\big)\big)\big|_U(D)}{z-z_0}$$

$$=\left(\beta\circ\alpha T^{-1}\left(\frac{F(z)-F(z_0)}{z-z_0}\right)\right)\bigg|_U(D) \to \beta\circ\alpha T^{-1}(F(z_0))|_U(D) \quad \text{as } z\to z_0.$$

Defining $g\colon \Omega\ni z\mapsto \tilde{f}_z(D)\in L(H)$ and $h\colon \Omega\ni z\mapsto K_z\in L(H)$, we get the desired decomposition, and the proof of case (a) is complete.

(b) Let $\Omega$ be an arbitrary open set.

In this case, functions $g$ and $h$ can be defined in each connected component, respectively. It should be mentioned that in this situation we shall not obtain the set $g(\Omega)$ with one (common for all its elements) set of eigenvectors.

### *REFERENCES*


[1] L. G. Brown, R. G. Douglas and P. A. Fillmore, *Unitary equivalence modulo compact operators and extensions of C*-algebras*, Lecture Notes in Mathematics 345, Springer-Verlag, 1973, p. 58-129.

[2] L. A. Coburn, *The Weyl theorem for non-normal operators*, The Michigan Mathematical Journal 13 (1966), p. 285-288.

[3] J. Globevnik and I. Vidav, *A note on normal operator-valued analytic functions*, Proceedings of the American Mathematical Society 37 (1973), p. 619-621.



INSTITUTE OF PHILOSOPHY
JAGIELLONIAN UNIVERSITY
KRAKÓW


*Reçu par la Rédaction le 21. 3. 1980*