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CONSTRUCTION OF A SCHAUDER BASIS IN SOME SPACES OF HOLOMORPHIC FUNCTIONS IN THE UNIT DISC

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1. INTRODUCTION

A sequence of vectors $\varphi_1, \varphi_2, \ldots$ in a separable Banach space B is said to be a *Schauder basis* for the space B if every vector φ of B can be expressed as a sum,

$$\varphi = \sum_k c_k \varphi_k,$$

convergent in B, with scalar coefficients c_k uniquely determined by φ. In this note we shall give an explicit construction of a Schauder basis in the following Banach spaces.

I. The spaces $H^p, 1 \leqslant p < \infty$. Recall that H^p consists of functions $\varphi(z)$ holomorphic in the unit disc $|z| < 1$ such that

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |\varphi(re^{i\vartheta})|^p d\vartheta$$

remains bounded for $0 \leqslant r < 1$; the norm is

$$\|\varphi\|_p = \lim_{r\to 1} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} |\varphi(re^{i\vartheta})|^p d\vartheta \right)^{1/p}.$$

This norm is equivalent with the L^p-norm of the boundary function $\varphi(\vartheta) = \lim_{r\to 1} \varphi(re^{i\vartheta})$ [1].

II. The space A. This space consists of all functions continuous in the closed unit disc $|z| \leqslant 1$ and holomorphic in the interior $|z| < 1$ with the uniform norm.

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[1] For $1 < p < \infty$, the trigonometric exponentials $1, e^{i\vartheta}, e^{2i\vartheta}, \ldots$ form a Schauder basis in H^p.

The question as to the existence of a Schauder basis in the space A occurs in Banach's treatise, *Théorie des opérations linéaires* (Warszawa 1932, p. 238).

Our procedure of construction is to form an interpolation series of a type studied in a recent paper [1] by choosing the nodes of interpolation conveniently in the unit disc. Actually, we take these nodes uniformly distributed on the circle $|z| = \frac{1}{2}$. We prove three things: 1° the sequence of remainders of our interpolation series interpolating a fixed function $\varphi \in B$, has a compactness property in the Banach space B; 2° this sequence of remainders converges to 0 uniformly on compact subsets of $|z| < 1$; 3° the coefficients of the series are uniquely determined by the given function φ.

It is curious that a dyadic partitioning occurs in the present note as well as in Schauder's original construction using Haar's functions [4], and in extensions of this construction [2].

An open question is to find out whether and how strongly the nodes can cluster at the unit circumference and still preserve the basis property of the functions $B_n(z)$.

2. AN INTERPOLATION SERIES

It is important for the sequel to enumerate a sequence $z_1, z_2, \ldots$ of distinct complex numbers of modulus $\frac{1}{2}$ in a certain way. We take the arguments of the z_j to lie in the interval $]-\pi, \pi]$ and to have the form

$$\frac{k}{2^n}\pi, \tag{2.1}$$

for some integers k, n. The rule of enumeration is to choose $z_1 = \frac{1}{2}$, then successively $z_2, z_3, \ldots$ such that the product of the distances to the already chosen points is maximum. In cases of ambiguity, when the maximum is realised at several distinct points, we choose the point with smallest positive argument of the form (2.1). The distribution of the points $z_1, z_2, \ldots, z_\nu$ remains for every index ν as near as possible to the uniform distribution of ν points on $|z| = \frac{1}{2}$. The sequence $z_1, z_2, \ldots$ thus begins as

$$\frac{1}{2}, \frac{1}{2}e^{i\pi}, \frac{1}{2}e^{i(\pi/2)}, \frac{1}{2}e^{-i(\pi/2)}, \frac{1}{2}e^{i(\pi/4)}, \frac{1}{2}e^{-i(3\pi/4)}, \frac{1}{2}e^{i(3\pi/4)}, \frac{1}{2}e^{-i(\pi/4)}, \frac{1}{2}e^{i(\pi/8)}, \ldots$$

Having fixed this sequence, put

$$B_0(z) = 1, \qquad B_n(z) = \prod_{k=0}^{n} \frac{\bar{z}_k}{|z_k|}\frac{z_k - z}{1 - \bar{z}_k z}.$$

Our interpolation series is a series of the form

$$\sum_{n=0}^{\infty} c_n B_n(z),$$

where the c_n are certain continuous linear forms on the Banach space B.

A sequence of functions $\gamma_0(t), \gamma_1(t), \ldots$ is uniquely determined on the unit circle $|t| = 1$ as a sequence of linear combinations of $1/(t-z_k)$ by the triangular systems

$$\gamma_0(t)B_0(z_k)+\ldots+\gamma_n(t)B_n(z_k) = \frac{1}{t-z_k} \quad (k = 1, 2, \ldots, n+1).$$

The following identity in t, z can be checked, for instance by examining the zeros and poles on both sides:

$$\frac{1}{t-z} - \sum_{k=0}^{n} \gamma_k(t)B_k(z) = \frac{B_n(z)(z-z_{n+1})}{B_n(t)(t-z_{n+1})}\cdot\frac{1}{t-z}.$$

Now if $\varphi(z)$ is any function belonging to a space B, it is represented by the Cauchy integral over its boundary values on $|t| = 1$. Hence, for $|z| < 1$ and $n = 0, 1, \ldots,$ we have

$$\varphi(z)-\sum_{k=0}^{n} c_k B_k(z) = \frac{1}{2\pi i}\int\limits_{|t|=1} \frac{B_n(z)(z-z_{n+1})}{B_n(t)(t-z_{n+1})}\,\frac{\varphi(t)}{t-z}\,dt, \tag{2.2}$$

where

$$c_k = \frac{1}{2\pi i}\int\limits_{|t|=1} \varphi(t)\gamma_k(t)\,dt.$$

Denote the right-hand side of (2.2) by $R_n(z) = R_n(z; \varphi)$. Clearly, if $\varphi \in B$, then also $R_n \in B$.

To show that $R_n(z)$ converges to 0 uniformly for $|z| \leqslant r < 1$, we use the estimate

$$\left|\frac{z-z_k}{1-\bar{z}_k z}\right| \leqslant \frac{1+2r}{2+r} \quad (|z| \leqslant r).$$

We then have

$$|R_n(z)| \leqslant \frac{1}{2\pi}\left(\frac{1+2r}{2+r}\right)^n\cdot\frac{1+2r}{1-r}\cdot\int\limits_{|t|=1} |\varphi(t)|\cdot|dt|,$$

which proves the stated convergence to 0.

3. A COMBINATORIAL LEMMA

By the n^{th} *generation* we understand the collection of those points z_j with argument of the form (2.1) for $k = -2^n+1, \ldots, 2^n-1, 2^n$. An interval of length p in the n^{th} generation is a sequence of p adjacent points on $|z| = \frac{1}{2}$ which belong to the n^{th} generation.

An important role is played by the following lemma of combinatorial nature:

LEMMA. *For every n, for every pair of disjoint intervals A_1, A_2 of equal length in the n^{th} generation and for every initial segment J of the sequence $z_1, z_2, \ldots$, if J intersects A_1 and A_2 in k_1 and k_2 points, then the excess $|k_1 - k_2|$ satisfies*

$$|k_1 - k_2| \leqslant 2.$$

Remark. In the sequel, any absolute constant would do as an upper bound for $|k_1 - k_2|$.

Proof. Let us begin by writing the points of the n^{th} generation in a table,

$$\begin{array}{cccc} z_1 & z_2 & z_3 & z_4 \\ z_5 & z_6 & z_7 & z_8 \\ z_9 & \ldots & & \\ & \ldots & & z_{2^n}, \end{array} \tag{3.1}$$

whereby the division into columns is nothing but the division according to quadrants on $|z| = 1/2$. It is enough to consider the intervals A_1, A_2 as lying in one and the same column. Since the argument of each point (3.1) is some absolute constant multiple of

$$\frac{\varepsilon_1}{2} + \frac{\varepsilon_2}{2^2} + \ldots + \frac{\varepsilon_n}{2^n}, \qquad \varepsilon_\nu = 0 \text{ or } 1,$$

we can use the usual binary notation $(\varepsilon_1 \varepsilon_2 \ldots \varepsilon_n)$ to represent such a point. Thus for instance for $n = 4$ we would have the column

ε_1	ε_2	ε_3	ε_4
0	0	0	0
1	0	0	0
0	1	0	0
1	1	0	0
0	0	1	0
1	0	1	0
0	1	1	0
1	1	1	0
0	0	0	1
1	0	0	1
0	1	0	1
1	1	0	1
0	0	1	1
1	0	1	1
0	1	1	1
1	1	1	1.

According to our rule of enumeration, the points of a column are enumerated according to height, highest first, etc. The points of the equal intervals A_1, A_2 lie scattered in a certain manner amongst the points of the column.

Proceed by induction on the generation index.

Direct examination shows that the maximum excess for arbitrary intervals A_1, A_2 of equal length in the 3^{rd} generation is 2. (There are no disjoint intervals containing at least 3 points in the 2^{nd} generation; the existence of such disjoint intervals is required for our induction.)

1° Suppose an excess of 3 is attained *before* passing the midpoint of the column of the n^{th} generation. Then, since each point in the upper half-column has the n^{th} digit $\varepsilon_n = 0$, we can consider the configuration (initial segment, A_1, A_2) restricted to the $(n-1)^{th}$ generation. However, in the $(n-1)^{th}$ generation it is impossible to have an excess of 3, by our induction assumption. 2° Suppose an excess of 3 is attained *after* traversing at least half of the column. We then consider the dual intervals A_1' and A_2' obtained by exchanging 0's and 1's in the binary expansion (reflection about mid-point), and we are in the impossible situation 1° just dealt with. This completes the proof of the lemma.

4. COMPACTNESS OF THE SEQUENCE OF REMAINDERS

We are going to examine the variation over the range $-\pi < \vartheta \leqslant \pi$ of the argument of the product

$$B_n(e^{i\vartheta}) = \exp\{i \sum_{k=1}^{n} \omega_k(\vartheta)\},$$

where $\omega_k(\vartheta) = \arg(z_k - e^{i\vartheta}) - \arg(z_k^* - e^{i\vartheta})$, $z_k^* = 1/\bar{z}_k$. In fact, we shall prove the following

LEMMA 2. *The sequence of functions*

$$a_n(\vartheta) = \sum_{k=1}^{n} \omega_k(\vartheta), \qquad n = 1, 2, \ldots,$$

forms an equicontinuous family in the uniform topology.

Proof. We shall show that the derivatives satisfy

$$\sup_{\vartheta, n} \left| \frac{da_n}{d\vartheta} \right| < \infty. \tag{4.1}$$

It results by a trivial computation that

$$\cos\omega_k(\vartheta) = \frac{4 - 5\cos(\vartheta - \vartheta_k)}{5 - 4\cos(\vartheta - \vartheta_k)}, \qquad z_k = \frac{1}{2} e^{i\vartheta_k}.$$

By differentiation we obtain

$$(4.2) \qquad \left|\frac{d\omega_k}{d\vartheta}\right| = \frac{3}{5-4\cos(\vartheta-\vartheta_k)},$$

where the sign depends upon the relative position of $e^{i\vartheta}$ and z_k. By (4.2) it follows that

$$(4.3) \qquad \sup_{\vartheta,n}\left|\frac{da_n}{d\vartheta}\right| \leqslant \sup_{\vartheta,m}\left|\frac{da_{2m}}{d\vartheta}\right| + 3,$$

so that it is sufficient to consider only indices n which are multiples of 2.

We shall now prove that

$$(4.4) \qquad \sup_{m,\theta}\left|\frac{da_{2m}}{d\vartheta}\right| < \infty,$$

and this will prove that the set of functions $B_\nu(e^{i\theta}), \nu = 0, 1, \ldots,$ is equicontinuous.

According to the rule of enumeration, m points amongst $z_1, z_2, \ldots, z_{2m}$ fall in every semi-circle of $|z| = \frac{1}{2}$ (counting end-points as lying on both semi-circles). Let n be the maximum index such that all points of the $(n-1)^{\text{st}}$ generation occur amongst $z_1, \ldots, z_{2m}$. Let (Σ_1, Σ_2) be any partition of $|z| = \frac{1}{2}$ into semicircles with end-points coinciding with points of the n^{th} generation. We are now in a position to pair the functions $\omega_k(\theta)$ associated with $z_k \in \Sigma_1$ with functions $\omega_j(\theta)$ associated with $z_j \in \Sigma_2$. It may not be always possible to associate with $z_k \in \Sigma_1$ its reflection in the diameter (Σ_1, Σ_2) because the reflected point may not belong to the segment $z_1, \ldots, z_{2m}$. We adopt the following rule of pairing. Traverse Σ_1 starting at one end-point and associate with the first point encountered which belongs to the sequence $z_1, \ldots, z_{2m}$ the first point of $z_1, \ldots, z_{2m}$ encountered by traversing Σ_2, starting at the same end-point. Delete the two associated points from $z_1, \ldots, z_{2m}$ and repeat until all points are paired. The lemma of § 3 shows that this will pair points whose arguments are out of kilter by at most

$$2\cdot\frac{1}{2^n}\cdot 2\pi \leqslant A/2^n,$$

where A is an absolute constant.

Let θ_0 be the argument of the end-point of Σ_1 and consider the interval $|\theta-\theta_0| \leqslant \pi/2^{n-1}$. Then, by (4.2),

$$\left|\frac{d\omega_k}{d\vartheta}+\frac{d\omega_j}{d\vartheta}\right|$$

$$= 3\left|\frac{1}{5-4\cos(\theta_0-\theta_k+\eta)} - \frac{1}{5-4\cos(\theta_0-\theta_k-\eta+\mu)}\right|, \qquad |\mu| \leqslant A/2^n,$$

where we have put $\eta = \theta - \theta_0$. It is essential to observe that the minus sign occurs on the right-hand side, since the increments of ω_k and ω_j are in opposite sense. It follows that

$$(4.5)\quad \left|\frac{da_{2m}}{d\theta}\right| = \left|\sum_{k,j}\left(\frac{d\omega_k}{d\theta}+\frac{d\omega_j}{d\theta}\right)\right| \leqslant 2^n O\left(\frac{1}{2^n}\right) = O(1), \qquad |\theta-\theta_0| \leqslant \pi/2^{n-1},$$

with an upper bound independent of θ_0, θ, n. Since the semi-circle Σ_1 is arbitrary, we can drop the condition $|\theta-\theta_0| \leqslant \pi/2^{n-1}$ in (4.5), and this yields (4.4), (4.1) and the equicontinuity of the sequence $B_n(e^{i\theta})$.

LEMMA 3. *The k^{th} Fourier coefficient of $B_n(e^{i\theta})$ in absolute value does not exceed $A/2^k$, where A denotes an absolute constant.*

Proof. Writing $z_j = 1/2e^{i\vartheta_j}$ and

$$S_j(\vartheta) = \sum_{k=1}^{\infty} \frac{e^{-ik\vartheta_j}}{2^k} e^{ik\vartheta},$$

we have

$$\frac{\bar{z}_j}{|z_j|}\,\frac{z_j - e^{i\vartheta}}{1-\bar{z}_j e^{i\vartheta}} = \frac{1}{2}\big(1-3S_j(\vartheta)\big)$$

and

$$B_n(e^{i\vartheta}) =$$

$$\left(\frac{1}{2}\right)^n \left\{1-3\sum_{j=1}^{n} S_j(\vartheta) + 3^2 \sum_{1\leqslant j<l\leqslant n} S_j(\vartheta)S_l(\vartheta) - \ldots + (-3)^n S_1(\vartheta)S_2(\vartheta)\ldots S_n(\vartheta)\right\}.$$

The k^{th} Fourier coefficient of $B_n(e^{i\vartheta})$ is therefore equal to

$$(4.6)\quad \left(\frac{1}{2}\right)^n \left\{-\frac{3}{2^k}\sum_{j=1}^{n} e^{-ik\vartheta_j} + \frac{3^2}{2^{2k}} \sum_{1\leqslant j<l\leqslant n}\ \sum_{\nu_j+\nu_l=k} e^{-i\nu_j\vartheta_j - i\nu_l\vartheta_l} - \right.$$

$$\left. - \ldots + \frac{(-3)^n}{2^{2nk}} \sum_{\nu_1+\ldots+\nu_n=k} e^{-i\nu_1\vartheta_1-\ldots-i\nu_n\vartheta_n}\right\}.$$

Using here the estimate

$$\left|\sum_{\nu_1+\ldots+\nu_m=k} e^{-i\nu_1\vartheta_1-\ldots-i\nu_m\vartheta_m}\right|$$

$$< 2k^{m-1}\sqrt{m} \int_{t_{m-1}=0}^{1} \ldots \int_{t_1=0}^{1-t_2-\ldots-t_{m-1}} dt_1 \ldots dt_{m-1} < 2k^{m-1},$$

we find that (4.6) does not exceed in absolute value

$$\frac{2}{k2^n}\left\{\binom{n}{1}(3k/2^k)+\binom{n}{2}(3k/2^k)^2+\ldots+(3k/2^k)^n\right\}$$

$$=\frac{2}{k2^n}\left\{\left(1+\frac{3k}{2^k}\right)^n-1\right\}$$

$$<\frac{A}{2^k},$$

as required.

With perhaps a different constant, the Fourier coefficients γ_{nk} of $\beta_n(t)=\big(B_n(t)(1-\bar{z}_{n+1}t)\big)^{-1}$ satisfy an estimate of the same type.

LEMMA 4. *For each function $\varphi \in A$ the sequence of functions*

$$C_n(z)=\frac{1}{2\pi i}\int\limits_{|t|=1}\frac{\varphi(t)\beta_n(t)}{t-z}\,dt, \qquad n=1,2,\ldots,$$

is an equicontinuous set in the annulus $\frac{3}{4}\leqslant|z|\leqslant 1$.

Proof. For $\Delta=e^{i\delta}$, δ real, we have

$$C_n(\Delta z)=\frac{1}{2\pi i}\int\limits_{|t|=1}\frac{\varphi(\Delta t)\beta_n(\Delta t)}{t-z}\,dt$$

$$=\frac{1}{2\pi i}\int\limits_{|t|=1}\frac{\varphi(\Delta t)}{t-z}\sum_{k=0}^{\infty}\gamma_{nk}\Delta^{-k}t^{-k}\,dt$$

$$=\sum_{k=0}^{\infty}\gamma_{nk}\frac{1}{2\pi i}\int\limits_{|t|=1}\frac{\varphi(\Delta t)}{\Delta^k t^k(t-z)}\,dt$$

$$=\sum_{k=0}^{\infty}\gamma_{nk}\Delta^{-k}z^{-k}\left(\varphi(\Delta z)-\sum_{j=0}^{k-1}\frac{\varphi^{(j)}(0)}{j!}\Delta^j z^j\right). \tag{4.7}$$

In view of the trivial bound,

$$\left|\sum_{j=0}^{k-1}\frac{\varphi^{(j)}(0)}{j!}z^j\right|<\big((1+\sigma)|z|\big)^{k-1}, \qquad \text{any } \sigma>0,$$

it follows that the series (4.7) converges uniformly with respect to $|\delta|\leqslant 1$, $\frac{3}{4}\leqslant|z|\leqslant 1$, $n=1,2,\ldots$, provided only that $\sigma<\frac{1}{3}$. This suffices to establish Lemma 4.

The reader will note that the continuity of $\varphi(z)$ in the closed unit disc $|z| \leqslant 1$ is used in an essential way at this point. Our argument fails, as it must, for $\varphi \in H^{\infty}$, where in fact no Schauder basis exists since the space H^{∞} is not separable.

LEMMA 5. *For each function $\varphi \in A$, the sequence of remainders $R_n(z;\varphi)$, $n = 1, 2, \ldots$, is an equicontinuous set in the space A.*

Proof. The remainders are of the form

$$R_n(z;\varphi) = \frac{1}{2\pi i} \int_{|t|=1} \frac{B_n(z)(z-z_{n+1})}{B_n(t)(t-z_{n+1})} \frac{\varphi(t)}{t-z}\, dt,$$

and we have $R_n \in A$. According to the maximum principle, it is sufficient to show that if $\varepsilon > 0$ is given, then

$$\sup_{\substack{n=1,2,\ldots \\ 3/4 \leqslant |z| \leqslant 1}} |R_n(\Delta z;\varphi) - R_n(z;\varphi)| < \varepsilon,$$

provided only that $\Delta = e^{i\delta}$ is sufficiently close to 1. As we have shown that the factors $B_n(z)(z-z_{n+1})$ and $C_n(z)$ do have this equicontinuity, it follows immediately that also $R_n(z;\varphi)$ does.

That the functions $B_n(z)$ form a Schauder basis in A now follows easily. Indeed, given $\varphi \in A$, there exists, according to Lemma 5, a subsequence $R_{n_k}(z;\varphi)$ convergent in A to some $\psi \in A$. By the Cauchy integral formula, $R_{n_k}(z;\varphi)$ converges uniformly on compact sets of $|z| < 1$ to ψ. But we know that $R_{n_k}(z;\varphi)$ converges uniformly on such sets to 0. Therefore every convergent subsequence of R_n converges to 0 in A, which implies that R_n converges to 0 in A.

It remains to verify the uniqueness of the coefficients of our interpolation series. Since the linear form given by evaluation,

$$\varphi \to \varphi(z_k),$$

is continuous on A, if we have

$$0 = \sum_{0}^{\infty} c_n B_n$$

in A, we can evaluate the series at points of the sequence $z_1, z_2, \ldots$ to conclude that the coefficients c_n must all vanish.

Only the slightest modification of the preceding line of reasoning will show that our construction also yields a Schauder basis in the spaces H^p, $1 \leqslant p < \infty$. We ought here to recall the following criterion of compactness in H^p, $1 \leqslant p < \infty$, due to Marcel Riesz (see [3]): *A subset $Q \subset H^p$ is (relatively) compact if and only if*

$$\sup_{\varphi \in Q} \int_{-\pi}^{\pi} |\varphi(e^{i\vartheta})|^p d\vartheta < \infty$$

and for every $\varepsilon > 0$ there is a $\delta > 0$ such that

$$\sup_{\varphi \in Q} \int_{-\pi}^{\pi} |\varphi(e^{i(\vartheta+h)}) - \varphi(e^{i\vartheta})|^p d\vartheta < \varepsilon,$$

provided only that $|h| < \delta$. This condition is satisfied for any uniformly equicontinuous family, and hence by the sequence $\{B_n(e^{i\vartheta})\}$. The proof of Lemma 4 is also valid for H^p, $1 \leqslant p < \infty$, and this is all that we need.

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