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ON THE RADIUS OF UNIVALENCE OF CERTAIN ANALYTIC FUNCTIONS

BY

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1. Introduction. Let (S) denote the class of functions analytic and univalent in the unit disc E ($|z| < 1$) and of the form

$$F(z) = z + a_2 z^2 + \ldots$$

By a *convex function* $F(z)$ we will mean $F(z) \in (S)$ whose range is a convex set. Such a function satisfies the differential inequality

$$\operatorname{Re}\{zF''(z)/F'(z) + 1\} > 0 \quad \text{for all } z \text{ in } E.$$

Denote this subclass by (K). By a *starlike function* $F(z)$ we will mean $F(z) \in (S)$ whose range is a starlike set with respect to the origin. Such a function satisfies the differential inequality

$$\operatorname{Re}\{zF'(z)/F(z)\} > 0 \quad \text{for all } z \text{ in } E.$$

Denote this subclass by (S^*). A trivial consequence of the above-mentioned is that a function $F(z)$ is convex if and only if $zF'(z)$ is starlike. By a *close-to-convex function* $F(z)$ we will mean $F(z) \in (S)$ whose range is close-to-convex set. Denote this class by (C). If $F(z) \in (C)$, then there exists a $g(z) \in (S^*)$ such that

$$\operatorname{Re}\{zF'(z)/g(z)\} > 0 \quad \text{for all } z \text{ in } E.$$

The class (C) has been introduced by Kaplan [1]. Note that

$$(K) \subset (S^*) \subset (C) \subset (S).$$

Recently Libera [2] proved that if $f(z)$ is a member of (C), (S^*) or (K), then

$$F(z) = (2/z) \int_0^z f(t)\,dt$$

belongs to (C), (S^*) or (K), respectively. Livingston [3] has studied the converse problem and obtained sharp results. Specifically, he has shown that if $F(z)$ is a member of (C), (S^*) or (K), then $f(z) = \frac{1}{2}(zF(z))'$ is a member of (C), (S^*) or (K) for $|z| < r_1 = \frac{1}{2}$, respectively.

The object of this paper is to improve upon most of Livingston's results by involving the second coefficient in the power expansion of $F(z)$. All results are sharp.

2. We need the following lemmas:

LEMMA 1. *Let $\varphi(z) = b_1 + b_2 z + \ldots$ be an analytic map of the unit disc E into itself. Then*

$$|\varphi(z)| \leqslant (|b_1| + |z|)/(1 + |b_1|\,|z|) \quad \text{for all } z \text{ in } E.$$

This lemma may be found in [4], p. 167.

LEMMA 2. *Let $\omega(z) = z\varphi(z) = b_1 z + b_2 z^2 + \ldots$ be an analytic map of the unit disc E into itself. Then*

$$|\omega'(re^{i\theta})| \leqslant \big(r + |\varphi(re^{i\theta})|\big)\big(1 - r|\varphi(re^{i\theta})|\big)/(1 - r^2).$$

Proof. Since $\omega(z) = z\varphi(z)$, $\omega'(z) = z\varphi'(z) + \varphi(z)$, and so

$$|\omega'(z)| \leqslant r|\varphi'(z)| + |\varphi(z)|, \qquad |z| = r.$$

Applying the maximum principle, one can easily show that $|\varphi(z)| < 1$ in the unit disc E. Hence (see [4], p. 168)

$$|\varphi'(z)| \leqslant \big(1 - |\varphi(z)|^2\big)/(1 - r^2) \quad \text{for all } z = re^{i\theta} \text{ in } E.$$

Substituting this estimate, we obtain

$$\begin{aligned}|\omega'(z)| &\leqslant r\big(1 - |\varphi(z)|^2\big)/(1 - r^2) + |\varphi(z)| \\ &= \big(r + |\varphi(z)|\big)\big(1 - r|\varphi(z)|\big)/(1 - r^2),\end{aligned}$$

valid for $r < 1$. The proof of the lemma is complete.

Remark. We remark in passing that $r + |\varphi(z)| \leqslant 1 + r|\varphi(z)|$ for all $r < 1$ and $|\varphi(z)| < 1$. Thus

$$\begin{aligned}|\omega'(z)| &\leqslant \big(r + |\varphi(z)|\big)\big(1 - |\omega(z)|\big)/(1 - r^2) \\ &\leqslant \big(1 - |\omega(z)|^2\big)/(1 - r^2).\end{aligned}$$

The right-hand side of this inequality is the well-known estimate for the derivative of the function of bound one in the unit disc. Lemma 2 is, therefore, an improvement on this known estimate when $\omega(0) = 0$.

With no loss of generality, we will replace the second coefficient a_2 in the power expansion of $F(z)$ by $|a_2|$. We proceed to prove the following theorem:

THEOREM 1. *Let $F(z) = z + 2pz^2 + \ldots$ be a member of the class (S^*), $f(z) = \frac{1}{2}(zF(z))'$. Then $p \leqslant 1$, and $f(z)$ is starlike for $|z| < r_p$, where r_p is the least positive root satisfying $2pr^3 + 3r^2 - 1 = 0$. This result is sharp.*

Proof. Since $F\in(S^*)$, it is well known that $2p\leqslant 2$ or $p\leqslant 1$. We also have $\mathrm{Re}\{zF'(z)/F(z)\}>0$. Consequently, there exists a function $\omega(z)=-pz+\ldots$, $|\omega(z)|<1$, regular in E such that

$$zF'(z)/F(z)=\Big(zf(z)-\int_0^z f(t)\,dt\Big)\Big/\int_0^z f(t)\,dt=\big(1-\omega(z)\big)/\big(1+\omega(z)\big)\quad\text{for } |z|<1.$$

Solving for $f(z)$, we have

$$f(z)=\Big(2\Big/\big(z(1+\omega(z))\big)\Big)\int_0^z f(t)\,dt.$$

A computation yields

$$zf'(z)/f(z)=\frac{1-\omega(z)-z\omega'(z)}{1+\omega(z)}=\frac{1-\omega(z)}{1+\omega(z)}-\frac{z\omega'(z)}{1+\omega(z)}.$$

To show that $f(z)$ is starlike in $|z|<r_p$, we must show that

$$\mathrm{Re}\{zf'(z)/f(z)\}>0\quad\text{in } |z|<r_p.$$

This condition is equivalent to

$$\text{(1)}\quad \mathrm{Re}\big\{\big(1-\omega(z)\big)/\big(1+\omega(z)\big)\big\}>\mathrm{Re}\big\{z\omega'(z)/\big(1+\omega(z)\big)\big\}\quad\text{for } |z|<r_p.$$

On the left-hand side of (1) substitute

$$\mathrm{Re}\big\{\big(1-\omega(z)\big)/\big(1+\omega(z)\big)\big\}=\big(1-|\omega(z)|^2\big)/|1+\omega(z)|^2,$$

and on the right-hand side of (1) substitute

$$\mathrm{Re}\big\{z\omega'(z)/\big(1+\omega(z)\big)\big\}\leqslant\big|z\omega'(z)/\big(1+\omega(z)\big)\big|.$$

Inequality (1) will be satisfied if

$$\text{(2)}\quad \big(1-|\omega(z)|^2\big)/|1+\omega(z)|^2>|z|\,|\omega'(z)|/|1+\omega(z)|\quad\text{for } |z|<r_p.$$

From Lemma 2,

$$|\omega'(z)|\leqslant\big(r+|\varphi(z)|\big)\big(1-|\omega(z)|\big)/(1-r^2),$$

where $\omega(z)=z\varphi(z)=z(-p+\ldots)$, $|z|=r$. Hence inequality (2) will be satisfied if

$$\big(1-|\omega(z)|^2\big)/|1+\omega(z)|^2>r\big(r+|\varphi(z)|\big)\big(1-|\omega(z)|\big)/(1-r^2)|1+\omega(z)|$$
$$\text{for } |z|=r<r_p.$$

Simplification of this yields

$$\text{(3)}\quad (1-r^2)/r>r+|\varphi(z)|.$$

But Lemma 1 gives the estimate $|\varphi(z)|\leqslant(p+r)/(1+rp)$, so inequality (3) will be satisfied if

$$(1-r^2)/r>r+(p+r)/(1+rp).$$

This reduces to $2pr^3+3r^2-1<0$, which gives the required root. When $p=1$, $2r^3+3r^2-1=0$ implies $r_1=1/2$ which is Livingston's. Simple analysis shows that, for $p<1$, $r_p>1/2$.

To see that the result is sharp, consider the function

$$F(z)=z/(1-2pz+z^2)=z+2pz^2+\ldots$$

On one hand, we have

$$zF'(z)/F(z)=(1-z^2)/(1-2pz+z^2)$$
$$=-1+2/\bigl(1+z(z-p)/(1-pz)\bigr)=\frac{1-u(z)}{1+u(z)},$$

where $u(z)=z(z-p)/(1-pz)$. Since $p\leqslant 1$,

$$|u(z)|=|z(z-p)/(1-pz)|\leqslant 1 \quad \text{for } |z|\leqslant 1.$$

This shows that

$$\operatorname{Re}\{zF'(z)/F(z)\}>0.$$

Thus $F\in(S^*)$. On the other hand,

$$f(z)=\tfrac{1}{2}\bigl(zF(z)\bigr)'=z(1-pz)/(1-2pz+z^2)^2.$$

Consequently,

$$zf'(z)/f(z)=(1-3z^2+2pz^3)/(1-pz)(1-2pz+z^2),$$

and thus $zf'(z)/f(z)=0$ for $z=-r_p$. The function $f(z)$ is, therefore, not starlike in any circle $|z|<r$ if $r>r_p$. This completes the proof of Theorem 1.

THEOREM 2. *Let* $F(z)=z+pz^2+\ldots$ *be a member of the class* (K), $f(z)=\frac{1}{2}\bigl(zF(z)\bigr)'$. *Then* $p\leqslant 1$, $f(z)$ *is univalent in* E *and is convex for* $|z|<r_p$, *where* r_p *is the same as in Theorem* 1. *This result is sharp.*

Proof. Since $F(z)$ is convex, it is well known that $p\leqslant 1$. Consider

$$zf'(z)/zF'(z)=f'(z)/F'(z)=1+\tfrac{1}{2}\bigl(zF''(z)/F'(z)\bigr).$$

It follows that $\operatorname{Re}\{zf'(z)/zF'(z)\}>1+\frac{1}{2}(-1)=\frac{1}{2}>0$ for z in E. Hence $f(z)$ is close-to-convex (relative to the starlike $zF'(z)$) and, therefore, is univalent in E.

To show that $f(z)$ is convex for $|z|<r_p$, we note that

$$zf'(z)=\tfrac{1}{2}\Bigl(z\bigl(zF'(z)\bigr)\Bigr)'. \tag{4}$$

Since $zF'(z)$ is starlike in E, Theorem 1 and (4) show that $zf'(z)$ is starlike for $|z|<r_p$, where r_p is the same as in Theorem 1. Thus $f(z)$ is convex for $|z|<r_p$.

This result is sharp for the function

$$F(z) = (\tfrac{1}{2}i\sin\theta)\log\big((1-ze^{-i\theta})/(1-ze^{i\theta})\big),$$

with $p = \cos\theta < 1$. For this function we get $zF'(z) = z/(1-2pz+z^2)$, $p < 1$. As for $p = 1$, we consider $F(z) = z/(1-z)$. In both cases, however, we get

$$zF'(z) = z/(1-2pz+z^2), \quad \text{where } p \leqslant 1.$$

The function $zF'(z)$, as is given above, is a member of the class (S^*) which has been verified in the proof of Theorem 1. Using (4) and a similar calculation to that performed in the proof of Theorem 1, one obtains

$$zf''(z)/f'(z)+1 = (1-3z^2+2pz^3)/(1-pz)(1-2pz+z^2).$$

Hence $zf''(z)/f'(z)+1 = 0$ for $z = -r_p$. This completes the proof of Theorem 2.

THEOREM 3. *Let* $F(z) = z+pz^2+\ldots$ *be analytic with* $\operatorname{Re}\{F'(z)\} > 0$ *for* $|z| < 1$, $f(z) = \frac{1}{2}\big(zF(z)\big)'$. *Then* $p \leqslant 1$, *and* $\operatorname{Re}\{f'(z)\} > 0$ *for* $|z| < r_p$, *where* r_p *is the smallest positive root satisfying*

$$r^4+3pr^3+2r^2-rp-1 = 0.$$

This result is sharp.

Proof. Let $F'(z) = \varrho(z) = 1+2pz+\ldots$, where $\operatorname{Re}\{\varrho(z)\} > 0$ for $|z| < 1$. Then it is well known that $2p \leqslant 2$ or $p \leqslant 1$ (see [4], p. 170). We also have

$$2f'(z) = 2F'(z)+zF''(z) = 2\varrho(z)+z\varrho'(z).$$

Hence, to show that $\operatorname{Re}\{f'(z)\} > 0$ for $|z| < r_p$ is equivalent to

$$2\operatorname{Re}\{\varrho(z)\} > |z\varrho'(z)| \quad \text{for } |z| < r_p. \tag{5}$$

Since $\operatorname{Re}\{\varrho(z)\} > 0$, there exists a regular function $\omega(z)$ in E such that $|\omega(z)| < 1$ and

$$\big(\varrho(z)-1\big)/\big(\varrho(z)+1\big) = \omega(z) = z\varphi(z) = pz+\ldots \tag{6}$$

It follows from (6) that

$$\varrho(z) = \big(1+\omega(z)\big)/\big(1-\omega(z)\big).$$

Thus

$$\varrho'(z) = 2\omega'(z)/\big(1-\omega(z)\big)^2.$$

From the estimate of Lemma 2 and the above-mentioned fact, we have

$$|\varrho'(z)| \leqslant 2\big(r+|\varphi(z)|\big)\big(1-|\omega(z)|\big)/(1-r^2)|1-\omega(z)|^2, \tag{7}$$

where $\varphi(z) = p + \ldots$ as in (6), $|z| = r$. On the left-hand side of (5) substitute

$$\operatorname{Re}\{\varrho(z)\} = \left(1-|\omega(z)|^2\right)/|1-\omega(z)|^2,$$

and on the right-hand side of (5) substitute the estimate in (7). Therefore, inequality (5) is satisfied if

$$(8) \quad 2\left(1-|\omega(z)|^2\right)/|1-\omega(z)|^2 > 2r\left(r+|\varphi(z)|\right)\left(1-|\omega(z)|\right)/(1-r^2)|1-\omega(z)|^2$$

is valid for $|z| < r_p$. Simplification of (8) yields

$$(1-r^2)/r > \left(r+|\varphi(z)|\right)/\left(1+r|\varphi(z)|\right).$$

Applying Lemma 1 to $\varphi(z) = p + \ldots$ in the above-mentioned inequality, we then have after simplification

$$(1-r^2)/r > (p+2r+pr^2)/(1+2rp+r^2).$$

This reduces to

$$r^4+3pr^3+2r^2-rp-1 < 0,$$

which gives the required root r_p.

When $p = 1$, $r^2+r-1 = 0$, thus $r_1 = \frac{1}{2}(5^{1/2}-1)$, which is Livingston's.

The result is sharp for $F(z)$ for which

$$F'(z) = (1-z^2)/(1-2pz+z^2) = 1+2pz+\ldots$$

For this function

$$\operatorname{Re}\{F'(z)\} = \operatorname{Re}\{(1-z^2)(1-2p\bar{z}+\bar{z}^2)/|1-2pz+z^2|^2\}.$$

Thus $\operatorname{Re}\{F'(z)\} > 0$ for $|z| < 1$ if

$$\operatorname{Re}\{(1-z^2)(1-2p\bar{z}+\bar{z}^2)\} > 0.$$

Let $x = \operatorname{Re}\{z\}$, $r = |z|$. Then the above-mentioned inequality is equivalent to

$$(9) \quad 1-r^4-2px(1-r^2) > (1-r^2)(1+r^2-2pr) > 0.$$

Let $g(r) = 1+r^2-2pr$. The function $g(r)$ attains a positive minimum at $r = p$ with $p < 1$. Thus (9) implies that $\operatorname{Re}\{F'(z)\} > 0$ for $|z| < 1$ and $p < 1$.

For $p = 1$, $F'(z) = (1+z)/(1-z)$, and, therefore, $\operatorname{Re}\{F'(z)\} > 0$ for $|z| < 1$. Thus $\operatorname{Re}\{F'(z)\} > 0$ for $|z| < 1$ and $p \leqslant 1$.

From $2f'(z) = 2F'(z)+zF''(z)$ we have, by substitution,

$$f'(z) = \frac{(1-z^2)(1-2pz+z^2)+z(p-2z+pz^2)}{(1-2pz+z^2)^2}.$$

We then have $f'(z) = 0$, when $1-pz-2z^2+3pz^3-z^4 = 0$ or $z = -r_p$. Thus $\operatorname{Re}\{f'(z)\} \ngtr 0$ in any circle $|z| < r$ if $r > r_p$. This completes the proof of Theorem 3.

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