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WHY CERTAIN ČECH-STONE REMAINDERS ARE NOT HOMOGENEOUS

BY

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All spaces under consideration are completely regular. If X is a space, then X^* denotes the Čech-Stone remainder $\beta X - X$ of X, and X^{**} denotes $(X^*)^*$.

The space Y is *homogeneous* if for any two points x and y of Y there is a homeomorphism h from Y onto itself with $h(x) = y$.

The space Y is *nowhere locally compact* if no point has a compact neighborhood.

1. Introduction. Frolík [9] has proved that X^* is not homogeneous if X is not pseudocompact (see also [8]). The proof is based on a cardinality argument, it does not show why X^* is not homogeneous, since it does not yield two points with different behavior. The purpose of this note is to give a proof of special cases of Frolík's theorem, which show why X^* is not homogeneous. This has been done previously under the assumption of additional set-theoretic axioms (see [1] and [12]), but we emphasize that we only use the axioms of ZFC. Our proof hinges on the following concept, related to but different from the remote points of Fine and Gillman [7] (see Remark 4.1).

Definition. If Y is a space, $p \in Y$ and $A \subseteq Y$, then p is *far* from A in Y if there is no closed discrete subset D of the subspace A such that $p \in \mathrm{Cl}_Y D$.

1.1. THEOREM. *Let X be a nowhere locally compact metrizable space. Then some but not all points of X^* are far from X^{**} in βX^*, and so X^* is not homogeneous.*

The key to this theorem is the following lemma:

1.2. LEMMA. *Let X be a non-compact metrizable space without isolated points. Then there is a point in X^* that is far from X in βX.*

It has been already known that this is true if X is the reals [7], but that proof does not work for the general case. The fact that the lemma holds if X is the rationals is of independent interest, and will be used in [2] and [5]. A totally different application of the lemma will be given in Remark 4.5.

We prove Theorem 1.1 from Lemma 1.2 in Section 2, prove Lemma 1.2 in Section 3, and collect some remarks in Section 4.

We use $\boldsymbol{N}$, $\boldsymbol{Q}$ and $\boldsymbol{R}$ for the positive integers, the rationals and the reals, respectively.

2. Far points and non-homogeneity. Theorem 1.1 is an immediate consequence of Lemma 1.2 and the following facts:

FACT 1. *Let h be any homeomorphism of Y onto itself. If $p \in Y$ is far from Y^*, then so is $h(p)$.*

Indeed, the Stone extension $\beta h : \beta Y \to \beta Y$ is a homeomorphism.

FACT 2. *If X is nowhere locally compact and not countably compact, then some point of X^* is not far from X^{**} in βX^*.*

Proof. Let $f: \beta X^* \to \beta X$ be the (unique) map with $f(x) = x$ for $x \in X^*$. Since X is nowhere locally compact, X^* is dense in βX and, consequently, f maps X^{**} onto X (see [6], p. 126, or [10], 6.11). Since continuous maps preserve countable compactness, X^{**} is not countably compact. The result follows.

FACT 3. *If X is nowhere locally compact, and some point of X^* is far from X in βX, then some point of X^* is far from X^{**} in βX^*.*

Proof. Let $f : \beta X^* \to \beta X$ be the unique map with $f(x) = x$ for $x \in X^*$. As above, f maps X^{**} onto X. Let $p \in X^*$ be not far from X^{**}. Then there is a closed discrete set D in the subspace X^{**} with $p \in \mathrm{Cl}_{\beta Y^*} D$. The restriction $f | X^{**}$ is a closed map from X^{**} onto X, since f is closed. Hence $f^{\to} D$ is closed in X. But, clearly, $p = f(p) \in \mathrm{Cl}_{\beta X} f^{\to} D$. Hence p is not far from X in βX.

3. Construction of far points. The construction of far points depends on the simple Lemma 3.1 below. We first introduce the notation

$$\mathscr{D}(X) = \{D \subseteq X: D \text{ is a closed discrete subset of } X\}.$$

3.1. LEMMA. *The following are equivalent for a normal space X:*

(a) *there is a point p in X^* which is far from X in βX;*

(b) *there is an open family $\mathscr{U} = \{U(D): D \in \mathscr{D}(X)\}$ in X with $D \subseteq U(D)$ for $D \in \mathscr{D}(X)$, such that no finite subfamily of $\mathscr{U}$ covers X;*

(c) *there is a closed family $\mathscr{F}$ in X with the finite intersection property, such that for each $D \in \mathscr{D}(X)$ there is an $F \in \mathscr{F}$ with $F \cap D = \emptyset$.*

Proof. (a) $\Rightarrow$ (b). For each $D \in \mathscr{D}(X)$ choose an open $U(D)$ in X with $p \notin \mathrm{Cl}_{\beta X} U(D)$ and $D \subseteq U(D)$.

(b) $\Rightarrow$ (c) is trivial.

(c) $\Rightarrow$ (a). Choose any $p \in \bigcap \{\mathrm{Cl}_{\beta X} F : F \in \mathscr{F}\}$. If $D \in \mathscr{D}(X)$, then there is an $F \in \mathscr{F}$ with $F \cap D = \emptyset$. Then $\mathrm{Cl}_{\beta X} F \cap \mathrm{Cl}_{\beta X} D = \emptyset$ since X is normal, whence $p \notin \mathrm{Cl}_{\beta X} D$.

Before we proceed to the formal proof of Lemma 1.2 we explain the idea of the proof. Let X be a non-compact metrizable space without isolated points. We want to construct $\mathscr{U} = \{U(D) \colon D \in \mathscr{D}(X)\}$ as in Lemma 3.1 (b). To this end we will find open $U(x, D)$ and $V(x, D)$ whenever $x \in D \in \mathscr{D}(X)$ such that

(A) $x \in U(x, D) \subset V(x, D)$;

(B) $V(x, D) \cap V(y, D) = \emptyset$ for distinct $x, y \in D$;

(C) $V(x, D) - U(x, D)$ is "big", in a sense to be made precise.

Then for $D \in \mathscr{D}(X)$ we put

$$U(D) = \bigcup \{U(x, D) \colon x \in D\}.$$

Let $\mathscr{F}$ be a finite subcollection of $\mathscr{D}(X)$, say $\mathscr{F}$ has m members. There should be a $G \in \mathscr{F}$ and a $g \in G$ such that $V(g, G) - U(g, G)$ (recall (C)) is not covered by

$$\{U(x, D) \colon x \in D \in \mathscr{F}, D \neq G\};$$

then, by (B), the $U(D)$'s, $D \in \mathscr{F}$, do not cover X. Since $\mathscr{F} - \{G\}$ has $m-1$ members, $V(g, G) - U(g, G)$ should, in some sense, have size m.

The major technical difficulty in a proof along these lines is the possibility that some $U(x, D)$, with $x \in D \in \mathscr{F} - \{G\}$, intersects $V(g, G) - U(g, G)$, even if $x \notin V(g, G)$. This difficulty could be avoided if X had a non-Archimedean base; recall the following definition:

Definition (cf. [11]). A base $\mathscr{B}$ for a space is called *non-Archimedean* if $A \subseteq B$ or $A \cap B = \emptyset$ or $A \supseteq B$ for all $A, B \in \mathscr{B}$.

This would put severe restrictions on X since it is clear that the members of a non-Archimedean base for a T_1-space are closed. Fortunately, there is an easy way out..

3.2. LEMMA. *If the normal space S contains a closed subspace T such that some point of T^* is far from T in βT, then some point of S^* is far from S in βS.*

This is a trivial consequence of Lemma 3.1 (c); of course, there also is an easy direct proof.

3.3. Proof of Lemma 1.2. Let X be a non-compact metrizable space; choose some compatible metric. The proof is divided into three steps.

Step 1. We construct a closed subspace Y of X, a cover $\{Y_m: m \in N\}$ and a non-Archimedean base $\mathscr{B} = \bigcup\{\mathscr{B}_n: n \in N\}$ of Y such that

(1) each $\mathscr{B}_n$ is a disjoint open cover of Y;

(2) $\mathscr{B}_1 = \{Y_m:\ m \in N\}$;

(3) each $A \in \mathscr{B}_{n+1}$ is included in some member of $\mathscr{B}_n$;

(4) if $B \in \mathscr{B}_n$ and $B \subseteq Y_m$, then B includes $m+1$ members of $\mathscr{B}_{m+1}$.

Since X is not compact, there is a discrete open family

$$\mathscr{W} = \{U_m:\ m \in N\}$$

consisting of non-empty open subsets of X, with $U_m \neq U_k$ if $m \neq k$. With an easy recursive construction one can find families $\mathscr{W}_n$, $n \geqslant 2$, consisting of non-empty open sets such that

(5) $\operatorname{diam} W < 1/n$ for $W \in \mathscr{W}_n, n \geqslant 2$;

(6) for each $W \in \mathscr{W}_{n+1}$ there is a $V \in \mathscr{W}_n$ with $\overline{W} \subseteq V, n \geqslant 1$;

(7) $\overline{V} \cap \overline{W} = \emptyset$ for distinct $V, W \in \mathscr{W}_n, n \geqslant 2$;

(8) for each $V \in \mathscr{W}_n$, if $V \subseteq Y_m$, then V includes exactly $m+1$ members of $\mathscr{W}_{n+1}, n \geqslant 1$.

By induction one shows that $\mathscr{W}_n$, $n \in N$, is a discrete family, hence, by (6),

$$\left(\bigcup \mathscr{W}_{n+1}\right)^- \subseteq \bigcup \mathscr{W}_n, \quad n \in N.$$

Therefore

$$Y = \bigcap_{n \in N}\left(\bigcup \mathscr{W}_n\right)$$

is closed. We put

$$Y_m = Y \cap U_m,\ m \in N, \qquad \text{and} \qquad \mathscr{B}_n = \{Y \cap W:\ W \in \mathscr{W}_n\},\ n \in N.$$

But then we have to make sure that no member of $\mathscr{B}_n$, $n \in N$, is empty. To this end we construct, simultaneously with the $\mathscr{W}_n$'s, subsets S_n of X such that the following holds for $n \in N$:

(9) each member of $\mathscr{W}_n$ contains exactly one point of S_n;

(10) each element of S_n is contained in a member of $\mathscr{W}_n$;

(11) $S_n \subseteq S_{n+1}$.

The "exactly one point" is essential if we construct $\mathscr{W}_{n+1}$, knowing $\mathscr{W}_n$, for it enables us to make sure that for each $V \in \mathscr{W}_n$ there is a $W \in \mathscr{W}_{n+1}$ with $\overline{W} \subset V$ which contains the unique point of $S_n \cap V$. Without this restriction we might not even be able to get $S_n \subseteq \bigcup \mathscr{W}_{n+1}$.

Step 2. For each $m \in N$ there is a family

$$\mathscr{U}_m = \{U_m(D):\ D \in \mathscr{D}(Y_m)\}$$

consisting of open subsets of the subspace Y_m, such that

(12) $\dot{D} \subseteq U_m(D)$ for $D \in \mathscr{D}(Y_m)$;

(13) Y_m is not the union of m members of $\mathscr{U}_m$.

Throughout the remaining part of this step we drop the subscript m in $\mathscr{U}_m$ and $U_m(D)$. Write $\mathscr{A}_n$ for $\{A \in \mathscr{B}_n : A \subseteq Y_m\}$; let

$$\mathscr{A} = \bigcup_{n \in N} \mathscr{A}_n.$$

We first discuss properties of $\mathscr{A}$ needed in the sequel. For each $A \in \mathscr{A}$ the smallest n with $A \in \mathscr{A}_n$ exists (in fact, n is unique), call it the ***level*** of A and denote by $\lambda(A)$. Then (1) and (3) imply

(14) for all $A, B \in \mathscr{A}$, if $\lambda(A) \geqslant \lambda(B)$, then either $A \subseteq B$ or $A \cap B = \emptyset$.

Also, by (1)-(3) there is for each $y \in Y_m$ and $n \in N$ a unique $A \in \mathscr{A}$ with $\lambda(A) = n$ and $y \in A$, denote it by $L(y, n)$. Then

(15) $\{L(y, n): n \in N\}$ is a neighborhood base at y for $y \in Y_m$.

Let $y \in D \in \mathscr{D}(Y_m)$; since D is discrete, (15) enables us to put

$$n(y, D) = \min\{k \in N: L(y, k) \cap D = \{y\}\}.$$

Next write

$$V(y, D) = L(y, n(y, D)), \quad U(y, D) = L(y, n(y, D)+1), \quad y \in D \in \mathscr{D}(Y_m);$$

$$U(D) = \bigcup \{U(y, D): y \in D\}, \quad D \in \mathscr{D}(Y_m);$$

$$\mathscr{U} = \{U(D): D \in \mathscr{D}(Y_m)\}.$$

We prove that this $\mathscr{U}$ satisfies (12) and (13). That (12) holds is clear. It follows from (14) that

(16) for all $D \in \mathscr{D}(Y_m)$, if $x, y \in D$ are distinct, then

$$V(x, D) \cap V(y, D) = \emptyset.$$

(Since $V(y, D) - U(y, D)$ has size m, by (4), we see that (A), (B) and (C) of the explanation of the proof hold.)

Let $\mathscr{F} \subseteq \mathscr{D}(Y_m)$ have cardinality less than or equal to m. Our plan is to construct, in at most m steps, an $A \in \mathscr{A}$ with $A \cap U(D) = \emptyset$ for all $D \in \mathscr{F}$, using (4) and (14); since $\emptyset \notin \mathscr{A}$, this will prove (13). To this end we introduce the following definition:

Let $A \in \mathscr{A}$ and $D \in \mathscr{D}(Y_m)$. Then we say that A *cuts* D provided that

(17) if $y \in D \cap A$, then $n(y, D) \geqslant \lambda(A)$;

(18) if $y \in \dot{D} - A$, then $U(y, D) \cap A = \emptyset$.

CLAIM. *If $K \in \mathscr{A}$, if $\mathscr{F} \subseteq \mathscr{D}(Y_m)$ has $1 \leqslant |\mathscr{F}| \leqslant m$, and if K cuts every member of $\mathscr{F}$, then there is a $K' \in \mathscr{A}$, and there is a proper subfamily $\mathscr{F}'$ of $\mathscr{F}$ such that*

(α) $K' \subseteq K$;

(β) $K' \cap U(D) = \emptyset$ *for* $D \in \mathscr{F} - \mathscr{F}'$;

(γ) K' *cuts every member of* $\mathscr{F}'$.

To see that (13) follows from the Claim, let $\mathscr{F} \subseteq \mathscr{D}(Y_m)$ have $1 \leqslant |\mathscr{F}| \leqslant m$. Then by at most m applications of the Claim we can find a (necessarily non-empty) $A \in \mathscr{A}$ with $A \cap U(D) = \emptyset$ for all $D \in \mathscr{F}$ (start with $K = Y_m$).

Proof of the Claim. Put

$$\mathscr{G} = \{D \in \mathscr{F}\colon D \cap K \neq \emptyset\}.$$

For $\mathscr{G} = \emptyset$ put $K' = K$, $\mathscr{F}' = \emptyset$. This works since, by (18), $U(D) \cap K = \emptyset$ for $D \in \mathscr{F} - \mathscr{G}$. Next assume that $\mathscr{G} \neq \emptyset$. Find $G \in \mathscr{G}$ and $g \in G \cap K$ such that

(19) $n(y, D) \geqslant n(g, G)$ whenever $D \in \mathscr{G}$ and $y \in D \cap K$.

By (16) there is, for each $D \in \mathscr{G}$, at most one $y \in D$ with $V(y, D) = V(g, G)$. Since $\lambda\big(V(y, D)\big) = n(y, D)$ whenever $y \in D \in \mathscr{D}(Y_m)$, it follows from (4), (19) and the fact that $|\mathscr{G}| \leqslant m$ that there is a $K' \in \mathscr{A}$ such that

(20) $K' \subseteq V(g, G)$;

(21) $\lambda(K') = n(g, G) + 1$;

(22) $n(y, D) > n(g, G)$ whenever $D \in \mathscr{G}$ and $y \in D \cap K'$.

Put $\mathscr{F}' = \mathscr{F} - \{G\}$. We will show that K' and $\mathscr{F}'$ work.

We check (α). Since $g \in K$, we have $n(g, G) \geqslant \lambda(K)$ by (19). Hence $V(g, G) \subseteq K$ by (14). Therefore, $K' \subseteq K$ by (20).

We check (β). $K' \subseteq V(g, G) - U(g, G)$ by (22). Hence $K' \cap U(G) = \emptyset$ by (14).

We check (γ). Let $y \in D \in \mathscr{F}$.

Case 1. $y \notin K$. Then $U(y, D) \cap K' = \emptyset$ by (α) and (18), since K cuts D.

Case 2. $y \in K - K'$. Then

$$\lambda\big(U(y, D)\big) = n(y, D) + 1 \geqslant n(g, G) + 1 = \lambda(K')$$

by (19) and (21). Hence $U(y, D) \cap K' = \emptyset$ by (14).

Case 3. $y \in K'$. Then

$$n(y, D) \geqslant n(g, G) + 1 = \lambda(K')$$

by (22) and (21).

This completes the proof of the Claim. Thus Step 2 is completed.

Step 3. There is a family $\mathscr{U} = \{U(D)\colon D \in \mathscr{D}(Y)\}$ consisting of open subsets of the subspace Y, such that

$D \subseteq U(D)$ for $D \in \mathscr{D}(Y)$;

$\mathscr{U}$ has no finite subcover.

Indeed, since $\{Y_m\colon m \in N\}$ is a pairwise disjoint family of open subspaces of Y, we can define the $U(D)$'s by

$$U(D) = \bigcup_{m \in N} U_m(D \cap Y_m).$$

Then, by Lemma 3.1 (b), there is a point in Y^* that is far from Y in βY. Hence, by Lemma 3.2, there is a point in X^* that is far from X in βX.

This completes the proof of the lemma.

4. Remarks. From now on, a point in X^* that is far from X in βX will be called simply a *far point*.

4.1. $p \in X^*$ is called a *remote point* in βX if p is not in the closure of a (not necessarily closed) discrete subset of X; for metrizable X this is equivalent to requiring that $p \notin \bar{A}$ for all nowhere dense A in X.

Remote points have been introduced by Fine and Gillman [7], who have shown that under CH βX has a remote point if $X = \boldsymbol{R}$ or $X = \boldsymbol{Q}$ and, in fact, if X is a non-compact separable metrizable space (cf. [7], Theorem 2.3). In [1] it is shown that this holds under Martin's Axiom, which is strictly weaker than CH.

Remote points and far points are not the same. Indeed, let A be a nowhere dense closed subspace of $\boldsymbol{Q}$ without isolated points. Then there is a far point p in $\boldsymbol{Q}^* \cap \bar{A}$.

4.2. In the special case $X = \boldsymbol{R}$ Eberlein has given a very simple proof of Lemma 1.2: Let μ be Lebesgue measure; then

$$\mathscr{U} = \{U \subseteq \boldsymbol{R}\colon\ U \text{ open},\ \mu(U) < \infty\}$$

satisfies Lemma 3.1 (b) (see [7], footnote 4). This argument also works if $X = \{\text{irrationals}\}$, or $X = \{\text{Sorgenfrey line}\}$, so for these X the proof that X^* is not homogeneous is particularly simple.

4.3. If Y is a space, $p \in Y$ and $A \subseteq Y$, call p *ω-far* from A in Y if there is no at most countable closed discrete D in the subspace A with $p \in \bar{D}$. It is clear from the argument in 3.1 that if X is not Lindelöf, then some point of X^* is ω-far from X in βX. Hence the ideas of Section 2 lead to

PROPOSITION. *If X is a normal nowhere locally compact space which is neither Lindelöf nor countably compact, then some but not all points of X^* are ω-far from X^{**}, and so X^* is not homogeneous.*

We do not know if normality is essential. A much more interesting question is whether the proposition holds for Lindelöf spaces. Since closed discrete sets in a Lindelöf space are at most countable, this leads to the following question:

QUESTION (**P 1066**). Let X be a non-compact Lindelöf space without isolated points. Is there an open family $\mathscr{U}$ which has no finite subcover yet for each closed discrete D in X there is a $U \in \mathscr{U}$ with $D \subseteq U$?

4.4. In [2] and [5] the following corollary to the proof of Lemma 1.2 will be used:

LEMMA. *There is a family $\mathcal{U}$ in Q such that*

(1) *for every $m \in N$ and finite $\mathcal{F} \subseteq \mathcal{U}$,*

$$Q - ([-m, m] \cup \bigcup \mathcal{F}) \neq \emptyset;$$

(2) *for every closed discrete $D \subset Q$ there is a $U \in \mathcal{U}$ with $D \subset U$.*

To show this, in the proof of Lemma 1.2 let $Y = X = Q$, and for $m \in N$ let

$$Y_m = \{q \in Q: m - \sqrt{2} < |q| < m+1-\sqrt{2}\}.$$

4.5. A totally different application of far points is the following. Let p be a far point in βQ, let T be the subspace $Q \cup \{p\}$ of βQ. Then there is no relatively discrete D in T which has p as a unique cluster point. It follows from [3] that T is not stratifiable, yet $T - \{p\}$ is stratifiable, being metrizable, and T is countable. Another example of this type has been given in [4].

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