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MAPPING ARCS AND DENDRITIC SPACES ONTO NETLIKE CONTINUA

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1. Introduction. The well-known Hahn-Mazurkiewicz theorem states that, for metric spaces, a continuum is the image of an arc under contin- uous map if and only if it is locally connected. Now, even for non-metric spaces, a Hausdorff image of an arc under a continuous map is locally connected, where by an *arc* is meant a continuum with only two non-cut points or, equivalently, a non-degenerate ordered continuum. This follows easily from the fact that a Hausdorff continuum is locally connected if and only if it satisfies Sierpiński's condition, which states that each open cover of the continuum has a finite refinement consisting of continua. For proofs of these results, see Kuratowski [2]. We also remark that Whyburn's proof of Kelley's theorem ([8], p. 39), which states that a Hausdorff image of a metric arc under a continuous map is arcwise connected, carries over *via* Zorn's lemma to the non-metric case with but little modification. That not every locally connected Hausdorff continuum is the image of an arc under a continuous map was first demonstrated by Mardešić [3], who gave an example of a locally connected Hausdorff continuum that is not arcwise connected. A conceptually simpler example has been given more recently by Cornette and Lehman [1]. For a discussion of early conjectures concerning the extension of the Hahn-Mazurkiewicz theorem to non-metric spaces, see Mardešić [3].

It is the main purpose of this paper to prove that every continuum in which each two points are separated by a finite point set is the image of an arc under a continuous map. An interesting question related to this theorem has been raised by Mardešić [4]. If the Hausdorff continuum M is the image of an arc under a continuous map and x and y are points of M contained in no metric subcontinuum of M, then are x and y separated in M by a finite point set? Thus, if the answer to Mardešić's question is in the affirmative, a partial converse of this theorem is true.

2. Definitions and preliminary theorems. A *dendritic space* is a connected space in which each two points are separated by a third point. A *netlike space* is a connected space in which each two points are separated by a finite point set. A space is *rim-finite* at the point p if each open set containing p contains an open set containing p with finite boundary. If ab is an arc from a to b and R is a space homeomorphic to $ab-\{b\}$ with non-cut point p, then R is called a *ray starting from* p. Netlike continua are known to be rim-finite, locally connected, and arcwise connected. These and other properties of netlike continua have been studied by Proizvolov [7]. The author proved in [6] that every dendritic continuum is the image of an arc under a continuous map, and in what follows that result is generalized to netlike continua.

THEOREM 1. *If R is a ray and f is a continuous one-to-one map of R into the netlike space Y, then there is a subray S of R such that $f(S)$ is a ray in Y.*

Proof. Suppose there exist two points x and y of R such that for each subray S of R both $f(x)$ and $f(y)$ are limit points of $f(S)$ in Y. Let F be a finite subset of Y separating $f(x)$ from $f(y)$ in Y. There is a subray S of R such that $S\cap f^{-1}(F)=\emptyset$. But then $f(S)\cup\{f(x),f(y)\}$ is a connected subset of $Y-F$. If there exists a point x of R such that, for each subray S of $R, f(x)$ is a limit point of $f(S)$ in Y, then let S be a subray of R starting from a point p of R such that $x\in R-S$. If no such point x exists, then let S be a subray of R starting from any point p of R. Therefore, the subspace $f(S)$ of Y is locally compact, connected, and each point of $f(S)-\{f(p)\}$ is a cut point of $f(S)$. It follows that $f(S)$ is a ray starting from $f(p)$.

THEOREM 2. *If R is a ray in the netlike continuum Y starting from the point x, then R has one and only one limit point y in $Y-R$ and $R\cup\{y\}$ is an arc from x to y.*

Proof. Since Y is compact and R is not compact, R has a limit point y in $Y-R$, and since Y is netlike, R has no other limit point in $Y-R$. Therefore, $R\cup\{y\}$ is closed in Y, and hence compact. Now, for each point z of $R-\{x\}$, z is a cut point of R, and since y is not a limit point of the arc xz of R, z is a cut point of $R\cup\{y\}$. It follows that $R\cup\{y\}$ is an arc from x to y.

THEOREM 3. *If X is a dense connected subspace of the dendritic continuum X' and f is a continuous one-to-one map of X into the netlike continuum Y, then f has a continuous extension to X'.*

Proof. It follows from Theorem 1 of [5] that both X and X' are uniquely arcwise connected. Let p be an end point of X'. For each x in $X'-X$ let px denote the arc in X' from p to x. Now, since X' is dendritic and X is a dense connected subset of X', $X'-X$ is totally disconnected.

Therefore, each open set in X' containing x contains a point w of X, and since the subarc pw of px is a subset of X, the ray $R_x = px - \{x\}$ is a subset of X. It follows from Theorems 1 and 2 that for each x in $X' - X$ there is a subray S_x of R_x such that $f(S_x)$ is a ray in Y with one and only one limit point in $Y - f(S_x)$. Let g be the extension of f to X' such that, for each x in $X' - X$, $g(x)$ is the limit point of $f(S_x)$ in $Y - f(S_x)$. We shall now prove that g is continuous. For each x in $X' - X$, g is continuous on S_x, $S_x \cup \{x\}$ is an arc, $g(S_x \cup \{x\})$ is an arc, and hence g is continuous on $S_x \cup \{x\}$. It follows that, for each x in $X' - X$, g is continuous on px. Now let $x \in X'$ and let u be a universal net in X' converging to x. Let q be an end point of X' such that x is a point of the arc pq. If u is eventually in pq, then $g \cdot u$ converges to $g(x)$, since g is continuous on pq. Suppose u is eventually in $X' - pq$. Let v be a subnet of u in $X' - pq$. Let r be the retraction of X' onto pq such that, for each z in $X' - pq$, $r(z)$ is the last point of the arc pz on pq. Now r is continuous, and hence the net $r \cdot v$ converges to $r(x) = x$. There is a subnet s of v with domain D such that $t = r \cdot s$ is a strictly monotone net in pq converging to x. For each a in D let $M_a = s_a t_a - \{s_a, t_a\}$. Then $M_a \subseteq X$ for each a in D, and since g is continuous on X, $g(M_a)$ is connected. Furthermore, if a and β are distinct elements of D, then M_a and M_β are disjoint, and since g is one-to-one on X, $g(M_a)$ and $g(M_\beta)$ are disjoint. Now $g \cdot s$ is a universal net in the compact space Y, and hence converges to a point y of Y, and since g is continuous on pq, $g \cdot t$ converges to $g(x)$. Each of y and $g(x)$ is a point of $\liminf\{g(M_a), a \in D\}$, and $\{g(M), a \in D\}$ has no finite subnet. Hence, if $y \neq g(x)$, then no finite subset of Y separates y from $g(x)$. Therefore $y = g(x)$, and hence $g \cdot u$ converges to $g(x)$. It follows that g is continuous.

THEOREM 4. *Every arcwise connected rim-finite space is semi-locally connected.*

Proof. Suppose p is a point of the arcwise connected rim-finite space X and U is an open set containing p. There is an open set V such that $p \in V \subseteq U$ and βV is finite. If C is a component of $X - V$ and $q \in C$, then there is a last point x of βV on some arc pq from p to q, and hence the ray $xq - \{x\}$ is a subset of C. Therefore, each component of $X - V$ has a boundary point in βV. Clearly, no point of βV is a boundary point of two components of $X - V$. It follows that $X - V$ has only finitely many components. Therefore, X is semi-locally connected at p.

Definitions. Let p be a point of the space Y and $X \subseteq Y$. Then p is said to be *accessible from* X if there is a subspace R of Y such that R is a ray starting from p and lying except for p in X. If Y is a space, $X \subseteq Y$, $p \in Y$, and R is a ray starting from p and lying except for p in X such that, for each point q of R, the arc pq of R has the same relative topology in both Y and R, then R is said to be *embedded in* X *at* p. The space R is said to be ***maximally embedded in X at p*** if R is embedded in X at p and is not a subspace of any other ray embedded in X at p. If p is a point of the dendritic space X and $\mathscr{S}$ is a collection of subspaces of X such that $X = \bigcup \mathscr{S}$, each element of $\mathscr{S}$ is a ray starting from p and, for each two elements H and K of $\mathscr{S}$, $H \cap K = \{p\}$, then X is called a ***simple tree growing from p***, and the elements of $\mathscr{S}$ are called the ***branches*** of X. If Y is a space, $X \subseteq Y$, $p \in Y$, and T is a simple tree growing from p such that each branch of T is embedded in X at p and the collection of all subsets U of T such that U contains p and is the intersection of T with an open set in Y is a base at p in the space T, then T is said to be ***embedded in X at p***. The space T is said to be ***maximally embedded in X at p*** if T is embedded in X at p and is not a subspace of any other simple tree embedded in X at p.

THEOREM 5. *If X is a subset of the space Y and p is a point of Y accessible from X, then there is a simple tree maximally embedded in X at p.*

Proof. Let $\mathscr{P}$ be the collection of all rays starting from p and lying except for p in X, order $\mathscr{P}$ by set inclusion, and let $\mathscr{S}$ be a maximal chain in $\mathscr{P}$. Then $B_0 = \bigcup \mathscr{S}$ is a ray maximally embedded in X at p. Suppose B_α has been defined for $\alpha < \beta$, and suppose p is accessible from $X - \bigcup_{\alpha<\beta} B_\alpha$. Let B_β be a ray maximally embedded in $X - \bigcup_{\alpha<\beta} B_\alpha$ at p. Thus, for some ordinal λ, B_α is defined for each $\alpha < \lambda$. Let

$$T = \bigcup_{\alpha<\lambda} B_\alpha.$$

For each subset U of T let U be open in T if and only if, for each branch B of T, $U \cap B$ is open in B. If $p \in U$, then there is an open set V in Y such that $V \cap T \subseteq U$. It follows that T is a simple tree maximally embedded in X at p.

Definitions. Suppose X is a set partially ordered by the relation $\leqslant$. For each x in X let

$$L(x) = \{y \mid y \leqslant x\}, \qquad DL(x) = \{y \mid y < x\},$$

$$U(x) = \{y \mid y \geqslant x\} \quad \text{and} \quad DU(x) = \{y \mid y > x\}.$$

These notations may be read: the lower set at x, the deleted lower set at x, the upper set at x and the deleted upper set at x, respectively. X is a ***tree*** if, for each x in X, $L(x)$ is fully ordered. X is a ***semilattice*** if each two elements of X have a greatest lower bound. The greatest lower bound of x and y is denoted by $x \wedge y$. A ***zero*** of X is an element 0 of X such that, for each x in X, $x \wedge 0 = 0$.

3. Proof of the main theorems. That every netlike continuum is the image of an arc under a continuous map follows from Theorem 4, other known theorems, and the following theorem, which is of some interest in its own right.

THEOREM 6. *Every netlike continuum is the image of an arcwise connected semi-locally connected dendritic space under a one-to-one continuous map.*

Proof. The idea of the proof is to decompose the netlike continuum into simple trees such that the resulting decomposition will have something like a dendritic structure. Indeed, the simple trees will play a role similar to that played by simple links or true cyclic elements in cyclic element theory.

Let Y be a netlike continuum. Well-order it and let p_0 be the first point of Y. Let T_0 be a simple tree maximally embedded in Y at p_0, let $\mathscr{S}_0 = \{T_0\}$, and let $M_0 = \bigcup \mathscr{S}_0$. Suppose M_α has been defined for $\alpha < \beta$. Let

$$H = \overline{\bigcup_{\alpha<\beta} M_\alpha},$$

let $p_{\beta 0}$ be the first point of H accessible from $Y-H$, and let $T_{\beta 0}$ be a simple tree maximally embedded in $Y-H$ at $p_{\beta 0}$. Suppose now that $p_{\beta m}$ and $T_{\beta m}$ have been defined for $m < n$. Let

$$K = \bigcup_{m<n} T_{\beta m},$$

let $p_{\beta n}$ be the first point of H accessible from $Y-(H \cup K)$, and let $T_{\beta n}$ be a simple tree maximally embedded in $Y-(H \cup K)$ at $p_{\beta n}$. Let $\mathscr{S}_\beta$ be the collection of all such sets $T_{\beta n}$, and let $M_\beta = H \cup \bigcup \mathscr{S}_\beta$. Now, for some ordinal λ, M_α is defined for each ordinal $\alpha < \lambda$. Let

$$M = \bigcup_{\alpha<\lambda} M_\alpha,$$

and suppose $\overline{M} \neq Y$. Since Y is arcwise connected, there exists an arc xy from a point x of $\overline{M}$ to a point y of Y such that $xy - \{x\} \subseteq Y - \overline{M}$. There is a smallest ordinal β such that $x \in M_\beta$. Then we infer, using the above-given notation, that $x = p_{\alpha n}$ for some n, and hence the simple tree $T_{\alpha n}$ is not maximally embedded in $Y-(H \cup K)$ at $p_{\beta n}$. Therefore $\overline{M} = Y$. Let

$$\mathscr{S} = \bigcup_{\alpha<\lambda} \mathscr{S}_\alpha \cup \big\{\{x\} \mid x \in Y - M\big\}.$$

Let $\{T_\alpha, \alpha \in D\}$ be a net of non-degenerate elements of $\mathscr{S}$, let p_α be the growth point of T_α for each α in D, and let $p \in Y$. Then $\{T_\alpha, \alpha \in D\}$ is said to *converge to* p if the net $\{p_\alpha, \alpha \in D\}$ converges to p.

If $\gamma \leqslant \lambda+1$, then a *γ-chain* is an indexed collection $\{T_\alpha | \alpha < \gamma\}$ such that

(1) for each $\alpha < \gamma$, $T_\alpha \in \mathscr{S}_\alpha$,

(2) for each α such that $\alpha+1 < \gamma$, $T_{\alpha+1}$ grows from a point of T_α, and

(3) if β is a limit ordinal less than γ, then the net $\{T_\alpha, \alpha < \beta\}$ converges to a growth point of T_β.

If $\mathscr{P} = \{T_\alpha, \alpha < \gamma\}$ is a γ-chain and $\mathscr{P}' = \{T_\alpha | \alpha < \gamma+1\}$ is a $(\gamma+1)$-chain, then $\mathscr{P}'$ is denoted by $\mathscr{P} \cup \{T_{\gamma+1}\}$, and $\mathscr{P}$ is said to be *attached* to $T_{\gamma+1}$. If $\mathscr{P} = \{T_\alpha | \alpha < \gamma\}$ is a γ-chain, $0 < \beta < \gamma$, and $\mathscr{P}' = \{T_\alpha | \alpha < \beta\}$ is a β-chain, then $\mathscr{P}'$ is called a *β-subchain* of $\mathscr{P}$.

We now define an *admissible chain* and a *choice function* ξ on a subfamily of $\mathscr{S}$ as follows:

(1) $\{T_0\}$ is an admissible 1-chain,

(2) if $\alpha > 0$, $T \in \mathscr{S}_\alpha$, and $\xi(T)$ is defined, then $\xi(T)$ is an admissible α-chain,

(3) if $\alpha > 0$, $T \in \mathscr{S}_\alpha$, and there is an admissible α-chain attached to T, then $\xi(T)$ is defined and $\xi(T) \cup \{T\}$ is an admissible $(\alpha+1)$-chain,

(4) for a limit ordinal β, a β-chain $\mathscr{P}$ is admissible if and only if, for each α such that $0 < \alpha < \beta$, the α-subchain of $\mathscr{P}$ is admissible, and

(5) if $p \in X - M$, β is a limit ordinal, and there is an admissible β-chain converging to p, then $\xi(\{p\})$ is defined and $\xi(\{p\}) \cup \{\{p\}\}$ is an admissible chain.

The elements of a chain will be called its *links*. Note that each chain has T_0 as a first link, but that it need not have a last link, even if it is a maximal admissible chain. Let $\mathscr{I}$ be the union of all admissible chains, and let $X = \bigcup \mathscr{I}$. We shall eventually prove that $\mathscr{I} = \mathscr{S}$ and $X = Y$.

Let T be a simple tree in $\mathscr{I}$ with growth point p. Then T has a natural partial order defined by $x \leqslant y$ if and only if $x = p$, $x = y$, or x separates p from y in T. Note that T is a semilattice tree with p as a zero element. Now $\mathscr{I}$ has partial order defined by $S \leqslant T$ if and only if S belongs to an admissible chain with last link T. To prove that $\mathscr{I}$ is partially ordered by $\leqslant$, note that each subchain of an admissible chain is admissible and each element of $\mathscr{I}$ is the last link of one and only one admissible chain. Finally, X has a partial order defined by $x \leqslant y$ if and only if

(1) there exists a degenerate element $\{p\}$ of $\mathscr{I}$ such that $x = y = p$,

(2) there exists a simple tree T in $\mathscr{I}$ such that $x, y \in T$ and $x \leqslant y$ in T, or

(3) there exist elements S and T of $\mathscr{I}$ such that $x \in S$, $y \in T$, $S \leqslant T$ in $\mathscr{I}$, and if $S < R \leqslant T$ in $\mathscr{I}$ and R grows from a point p of S, then $x \leqslant p$ in R.

The proof that X is partially ordered involves several cases and is omitted. A new topology for X is now defined in terms of the partial order on X and the topology on Y. Let a subbase for the new topology be the collection of all subsets U of X such that, for some open set V in Y and some point x in X, $U = (V \cap X) - U(x)$. The following lemmas are needed to prove that X is an arcwise connected rim-finite dendritic space in its new topology.

LEMMA 1. *$\mathscr{I}$ is a semilattice tree with zero.*

Proof. T_0 is clearly the zero element of $\mathscr{I}$. For each element T of $\mathscr{I}$, $L(T)$ is the admissible chain from T_0 to T, and since admissible chains are well ordered, $\mathscr{I}$ is a tree. Suppose S and T are incomparable elements of $\mathscr{I}$. There exist a first element H of $L(S)-L(T)$ and a first element K of $L(T)-L(S)$. Hence $DL(H) = DL(K)$, and if $DL(H)$ does not have a last link, then it converges to a point p and both H and K grow from p, which contradicts the maximality of the embedding of the simple trees H and K. Therefore, $DL(H)$ has a last link G, and $G = H \wedge K = S \wedge T$. It follows that $\mathscr{I}$ is a semilattice.

LEMMA 2. *X is a semilattice tree with zero.*

Proof. p_0 is clearly the zero element of X. Suppose $z \in X$, and x and y are points of $DL(z)$. Let R, S and T be elements of $\mathscr{I}$ such that $x \in R$, $y \in S$ and $z \in T$. Hence $R, S \leqslant T$, and since $\mathscr{I}$ is a tree, either $R \leqslant S \leqslant T$ or $S \leqslant R \leqslant T$. Assume $R \leqslant S \leqslant T$. Now if $R = S$, then, since R is a tree, $x \leqslant y$ or $y \leqslant x$. Suppose $R < S$. Since $x < z$, there is a point p of R which is a growth point of some element U of $L(T)$ and $x \leqslant p$ in R. Then $U \in L(S)$, and it follows that $x \leqslant y$. Therefore X is a tree. Now suppose x and y are points of X. Let S and T be elements of $\mathscr{I}$ such that $x \in S$ and $y \in T$. Let $R = S \wedge T$. There are points p and q of R such that p is a growth point of some element of $L(S)$ and q is a growth point of some element of $L(T)$. Then $x \wedge y$ is the point $p \wedge q$ of the semilattice R. It follows that X is a semilattice.

LEMMA 3. *For each x in $X-\{p_0\}$, $L(x)$ is an arc in Y from p_0 to x.*

Proof. Suppose γ is an ordinal and x is a point of some element T of $\mathscr{S}_\gamma$. The proof is by induction on γ. If $\gamma = 0$, then x is a point of some branch B of T_0 and $L(x)$ is an arc in B from p_0 to x. Since B is embedded in Y at p_0, $L(x)$ is an arc in Y from p_0 to x. Suppose $\gamma > 0$ and, for each $\beta < \gamma$, each element S of $\mathscr{S}_\beta$ and each point p of $S-\{p_0\}$, $L(p)$ is an arc in Y from p_0 to p. First suppose $\gamma = \beta + 1$ for some ordinal β. There exist an element S of $\mathscr{S}_\beta$ and a point p of S such that T grows from p. $L(p)$ is an arc in Y from p_0 to p. If $x \neq p$, then there is an arc A in T from p to x, $L(p) \cup A = L(x)$, $L(p) \cap A = \{p\}$, and hence $L(x)$ is an arc in Y from p_0 to x. Now suppose γ is a limit ordinal. Let $L(T) = \{T_\alpha \mid \alpha \leqslant \gamma\}$ and, for each $\alpha \leqslant \gamma$, let p_α be the growth point of T_α. For each α such that

$0 < \alpha < \gamma$, $L(p_\alpha)$ is an arc in Y from p_0 to p_α and if $\alpha < \beta < \gamma$, then $L(p_\alpha) \subset L(p_\beta)$. Let

$$R = \bigcup_{\alpha<\gamma} L(p_\alpha).$$

Then R is a ray in Y starting from p_0, and the net $\{p_\alpha, \alpha < \gamma\}$ converges to p_γ. It follows from Theorem 2 that $L(p_\gamma) = R\cup\{p_\gamma\}$ is an arc in Y from p_0 to p_γ, and it then follows as before that $L(x)$ is an arc in Y from p_0 to x. Finally, if $\{x\}$ is an end link of some admissible chain, then the proof is similar to the above.

LEMMA 4. *For each x in X, $L(x)$ has the same relative topology in both X and Y.*

Proof. We shall prove that, for each open set U in Y and each point y of Y, $(U-U(y))\cap L(x)$ is open in the subspace $L(x)$ of Y. If $y \in X - L(x)$, then $L(x)\cap U(y) = \emptyset$. If $y = x$, then $L(x)\cap U(y) = \{x\}$. If $y \in DL(x)$, then $L(x)\cap U(y) = L(x)-DL(y)$, and it follows from Lemma 3 that $L(x)-DL(y)$ is an arc in Y from x to y. Therefore, in any case, $L(x)\cap U(y)$ is closed in Y, and hence $(U-U(y))\cap L(x)$ is open in the subspace $L(x)$ of Y.

LEMMA 5. *For each x in X, $U(x)$ is closed in X and $DU(x)$ is open in X.*

Proof. $U(x)$ is closed in X, since $X - U(x)$ is open in X. Let $y \in DU(x)$. There is an open set V in Y containing y such that $\overline{V} \subseteq Y - L(x)$, and βV is finite. Let

$$(\beta V\cap X) - U(x) = \{x_1, \ldots, x_n\}.$$

Now, for each z in $(V\cap X) - U(x)$, $L(z)$ is an arc from p_0 to z, and hence, for some i, $x_i \in L(z)$ and $z \in U(x_i)$. Let

$$W = (V\cap X) - \bigcup_{i=1}^{n} U(x_i).$$

Then W is an open set in X, $y \in W$, and $W \subseteq DU(x)$. It follows that $DU(x)$ is open in X.

LEMMA 6. *X is dendritic in its new topology.*

Proof. Let x and y be points of X. If $x < y$, then there exists a point z such that $x < z < y$, and hence $X - \{z\}$ is the union of the two disjoint open sets $X - U(z)$ and $DU(z)$ containing x and y, respectively. Suppose x and y are incomparable. For each z in $DL(x) - L(x\wedge y)$ and w in $DL(y) - L(x\wedge y)$, $DU(z)$ and $DU(w)$ are disjoint open sets containing x and y, respectively. Let U denote the union of all such sets $DU(z)$, and let V denote the union of all such sets $DU(w)$. Then $X - \{x\wedge y\}$ is the union of the two disjoint open sets U and $V\cup(X - U(x\wedge y))$ containing x and y, respectively.

LEMMA 7. *X is rim-finite in its new topology.*

Proof. Let $x \in X$, let U be an open set in Y, and let $x_1, \ldots, x_n$ be points of X such that

$$(U \cap X) - \bigcup_{i=1}^{n} U(x_i)$$

is a basic open set in X containing x. There is an open set V in Y such that $x \in V \subseteq U$ and βV is finite. Let

$$W = (V \cap X) - \bigcup_{i=1}^{n} U(x_i).$$

Then $x \in W \subseteq U$ and W is open in X. For each subset M of X, let β_X denote the boundary of M in the space X. The following computation shows that $\beta_X W$ is finite:

$$\beta_X W \subseteq \bigcup_{i=1}^{n} \beta_X\big((V \cap X) - U(x_i)\big) \subseteq \bigcup_{i=1}^{n} \big(\beta_X(V \cap X) \cup \beta_X U(x_i)\big)$$

$$\subseteq \bigcup_{i=1}^{n} (\beta V \cup \{x_i\}) = \beta V \cup \{x_1, \ldots, x_n\}.$$

It now follows from the lemmas and Theorem 4 that X, in its new topology, is an arcwise connected semi-locally connected dendritic space. Furthermore, since the new topology of X is finer than the subspace topology of X, the inclusion map h of X in Y is continuous. It remains to be proved that $X = Y$. We first prove by induction that $\mathcal{S}_a \subseteq \mathcal{S}$ for each a. Clearly, $\mathcal{S}_0 \subseteq \mathcal{S}$. Suppose $\mathcal{S}_a \subseteq \mathcal{S}$ and $T \in \mathcal{S}_{a+1}$. Then T grows from some point of the last link of an admissible a-chain, and hence $\xi(T)$ is defined and $\xi(T) \cup \{T\}$ is admissible. Now suppose γ is a limit ordinal and $\mathcal{S}_a \subseteq \mathcal{S}$ for $a < \gamma$. Let $T \in \mathcal{S}_\gamma$ with growth point p. If $p \in X$, it follows, as in the proof of Lemma 3, that there is an admissible γ-chain converging to p, and hence $\xi(T)$ is defined and $\xi(T) \cup \{T\}$ is admissible. Suppose $p \in Y - X$. It follows from Theorem 3 of [5] that X is a dense subspace of some dendritic space X'. Now, from Theorem 3, the inclusion map h has a continuous extension g to X'. Since $g(X')$ is a compact subset of Y and X is dense in $g(X)$, $g(X') = \bar{X}$. There is an arc A in X' from p_0 to a point x of $g^{-1}(p)$ such that $A - \{x\} \subseteq X$. Now $g(A - \{x\})$ is a ray R in Y, $p \in Y - R$, and p is a limit point of R. It follows from Theorem 2 that $R \cup \{p\}$ is an arc $p_0 p$ in Y from p_0 to p. Again it follows, as in the proof of Lemma 3, that for each q in $p_0 p - \{p_0\}$ such that q is a growth point of some element T_q of $\mathcal{S}$ there is an admissible chain $\mathcal{P}_q$ converging to q. Since $L(q)$ is an arc, the growth point of each link of $\mathcal{P}_q$ is in $p_0 p$. Furthermore, if $r \in p_0 p - \{p_0\}$ and r is a growth point of some link of an admissible chain $\mathcal{P}_r$ converging to r, then one of the admissible chains $\mathcal{P}_q$ and $\mathcal{P}_r$ is a subchain of the other. Hence for some β the union $\mathcal{P}$ of all such admissible chains $\mathcal{P}_q$ is an admissible β-chain converging to a point y of $p_0 p$. Suppose $y \neq p$. Let z be a point of $p_0 p$ such that $y < z < p$. Then there is an element of $\mathcal{S}_\beta$ that grows from

y and contains some subarc of the arc yz, which contradicts the maximality of $\mathscr{P}$. Therefore $y = p$. Suppose $\beta \neq \gamma$. Since p is accessible from $Y - \overline{\bigcup_{a<\beta} M_a}$, there is an element S of $\mathscr{S}_\beta$ containing p, and since no two simple trees grow from the same point, p is not the growth point of S. Then T grows from a point of S, and hence $\gamma = \beta + 1$, which contradicts the supposition that γ is a limit ordinal. Therefore $\beta = \gamma$, and $\mathscr{P}$ is an admissible γ-chain converging to p. Hence $\xi(T)$ is defined, and $\xi(T) \cup \{T\}$ is admissible. It follows that $\mathscr{S}_a \subseteq \mathscr{I}$ for each a, and hence $\bar{X} = Y$. Finally, suppose $p \in Y - M$. It follows as before that there is an admissible chain converging to p, so that $\xi(\{p\})$ is defined and $\xi(\{p\}) \cup \{\{p\}\}$ is admissible. Therefore $X = Y$. This completes the proof of the theorem.

THEOREM 7. *Every netlike continuum is the image of an arc under a continuous map.*

Proof. Let Y be a netlike continuum. By Theorem 6, Y is the image of an arcwise connected semi-locally connected dendritic space X under a one-to-one continuous map f. By Theorem 3 of [5], X is a dense subspace of a dendritic continuum X'. By Theorem 3, f has a continuous extension g to X'. Finally, by Theorem 2 of [6], X' is the image of an arc A under a continuous map h. Therefore, $h \cdot g$ is a continuous map of A onto Y.

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