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HANKEL TRANSFORM ON LORENTZ SPACES

BY

LEONARDO COLZANI (COSENZA)

Let J_α, $\alpha \geq -1/2$, be the Bessel function of the first kind of order α. For suitable test functions on $\mathbf{R}_+$ one has Hankel's inversion formula

$$f(x) = \int_0^\infty \frac{J_\alpha(xy)}{(xy)^\alpha}\left\{\int_0^\infty \frac{J_\alpha(yt)}{(yt)^\alpha} f(t)t^{2\alpha+1}\,dt\right\}y^{2\alpha+1}\,dy.$$

Define the Rth partial sums of this integral by

$$S_R f(x) = \int_0^R \frac{J_\alpha(xy)}{(xy)^\alpha}\left\{\int_0^\infty \frac{J_\alpha(yt)}{(yt)^\alpha} f(t)t^{2\alpha+1}\,dt\right\}y^{2\alpha+1}\,dy,$$

and the maximal operator associated to the partial sums by

$$S^* f(x) = \sup_R |S_R f(x)|.$$

C. Herz (1954) showed that the partial sums converge in the norm of $L^p(\mathbf{R}_+, x^{2\alpha+1}\,dx)$ if and only if $\frac{4\alpha+4}{2\alpha+3} < p < \frac{4\alpha+4}{2\alpha+1}$, and E. Prestini (1988) obtained for the same range of p's the boundedness of the maximal operator. C. Kenig and P. Tomas (1980) showed that at the critical indexes the partial sum operators are not of weak type (p,p), but S. Chanillo (1984) showed that these operators are of restricted weak type (p,p).

In this paper we want to present a new and short proof of Chanillo's result, and at the same time to show that at the critical indexes the maximal operator is of restricted weak type. This of course implies the almost everywhere convergence of the partial sums of functions in the Lorentz spaces $L^{\frac{4\alpha+4}{2\alpha+3},1}(\mathbf{R}_+, x^{2\alpha+1}\,dx) + L^{\frac{4\alpha+4}{2\alpha+1},1}(\mathbf{R}_+, x^{2\alpha+1}\,dx)$.

Recall that the Lorentz spaces $L^{p,q}(\mathbf{R}_+, x^{2\alpha+1}\,dx)$ are defined by the (quasi-)norms

$$\|f\|_{p,q} = \left\{\frac{q}{p}\int_0^\infty [t^{1/p} f^*(t)]^q \frac{dt}{t}\right\}^{1/q},$$

where f^* is the non-increasing rearrangement of f.

THEOREM I. *If* $p = \frac{4\alpha+4}{2\alpha+3}$ *or if* $p = \frac{4\alpha+4}{2\alpha+1}$, *then the maximal operator* S^* *is bounded from* $L^{p,1}(\mathbf{R}_+, x^{2\alpha+1}dx)$ *into* $L^{p,\infty}(\mathbf{R}_+, x^{2\alpha+1}dx)$.

THEOREM II. (i) *The partial sum operators* $\{S_R\}$ *are not bounded from* $L^{\frac{4\alpha+4}{2\alpha+3},r}(\mathbf{R}_+, x^{2\alpha+1}\,dx)$, $1 < r \le \infty$, *into* $L^{\frac{4\alpha+4}{2\alpha+3},\infty}(\mathbf{R}_+, x^{2\alpha+1}\,dx)$.

(ii) *The operators* $\{S_R\}$ *are not defined from* $L^{\frac{4\alpha+4}{2\alpha+1},r}(\mathbf{R}_+, x^{2\alpha+1}\,dx)$, $1 < r \le \infty$, *into the space of tempered distributions.*

Proof of Theorem I. We first consider the case $p = \frac{4\alpha+4}{2\alpha+1}$. Then

$$\begin{aligned}
&S_R f(x) \\
&= \int_0^\infty (xy)^{-\alpha} \frac{RxJ_{\alpha+1}(Rx)J_\alpha(Ry) - RyJ_{\alpha+1}(Ry)J_\alpha(Rx)}{x^2 - y^2} f(y)y^{2\alpha+1}\,dy \\
&= \frac{1}{2}\int_0^\infty \frac{R(xy)^{-\alpha}J_{\alpha+1}(Rx)J_\alpha(Ry)}{x-y} f(y)y^{2\alpha+1}\,dy \\
&\quad - \frac{1}{2}\int_0^\infty \frac{R(xy)^{-\alpha}J_\alpha(Rx)J_{\alpha+1}(Ry)}{x-y} f(y)y^{2\alpha+1}\,dy \\
&\quad + \frac{1}{2}\int_0^\infty \frac{R(xy)^{-\alpha}J_{\alpha+1}(Rx)J_\alpha(Ry)}{x+y} f(y)y^{2\alpha+1}\,dy \\
&\quad + \frac{1}{2}\int_0^\infty \frac{R(xy)^{-\alpha}J_\alpha(Rx)J_{\alpha+1}(Ry)}{x+y} f(y)y^{2\alpha+1}\,dy .
\end{aligned}$$

These four terms are similar, and since the functions $\sqrt{t}J_\beta(t)$ are bounded, we are led to consider the operators

$$\begin{aligned}
T_R f(x) &= \int_0^\infty (xy)^{-\alpha-1/2} \frac{\sqrt{Ry}J_\alpha(Ry)}{x-y} f(y)y^{2\alpha+1}\,dy , \\
Zf(x) &= \int_0^\infty \frac{(xy)^{-\alpha-1/2}}{x+y} f(y)y^{2\alpha+1}\,dy .
\end{aligned}$$

The operator Z can be studied via the duality between the Lorentz spaces $L^{\frac{4\alpha+4}{2\alpha+1},1}(\mathbf{R}_+, x^{2\alpha+1}\,dx)$ and $L^{\frac{4\alpha+4}{2\alpha+3},\infty}(\mathbf{R}_+, x^{2\alpha+1}\,dx)$. Indeed, by the Hölder inequality for Lorentz spaces,

$$\begin{aligned}
|Zf(x)| &\le x^{-\alpha-1/2} \int_0^\infty y^{-\alpha-3/2}|f(y)|y^{2\alpha+1}\,dy \\
&\le c\|y^{-\alpha-3/2}\|_{\frac{4\alpha+4}{2\alpha+3},\infty}\|f\|_{\frac{4\alpha+4}{2\alpha+1},1}\, x^{-\alpha-1/2} .
\end{aligned}$$

Since $y^{-\alpha-3/2}$ is in $L^{\frac{4\alpha+4}{2\alpha+3},\infty}(\mathbf{R}_+, x^{2\alpha+1}\,dx)$ and $x^{-\alpha-1/2}$ is in $L^{\frac{4\alpha+4}{2\alpha+1},\infty}(\mathbf{R}_+, x^{2\alpha+1}\,dx)$, it follows that the operator Z is bounded from $L^{\frac{4\alpha+4}{2\alpha+1},1}(\mathbf{R}_+, x^{2\alpha+1}\,dx)$ into $L^{\frac{4\alpha+4}{2\alpha+1},\infty}(\mathbf{R}_+, x^{2\alpha+1}\,dx)$.

To study the operators $\{T_R\}$ decompose

$$\begin{aligned} T_R f(x) &= \int_0^\infty (xy)^{-\alpha-1/2} \frac{\sqrt{Ry}J_\alpha(Ry)}{x-y} f(y) y^{2\alpha+1}\,dy \\ &= \int_0^\infty \sqrt{Ry}J_\alpha(Ry)(xy)^{-\alpha} \frac{(xy)^{-1/2} - y^{-1}}{x-y} f(y) y^{2\alpha+1}\,dy \\ &\quad + \int_0^\infty (y/x)^\alpha \frac{\sqrt{Ry}J_\alpha(Ry)}{x-y} f(y)\,dy \\ &= H_R f(x) + K_R f(x)\,. \end{aligned}$$

This is the main idea of the paper. We have substituted the weight $(y/x)^{\alpha+1/2}$ in the definition of the operators $\{T_R\}$ with the weight $(y/x)^\alpha$ in the definition of $\{K_R\}$. This will allow us to apply the theory of A_p weights to $p = \frac{4\alpha+4}{2\alpha+1}$. The operators $\{H_R\}$ will be studied via the duality between $L^{\frac{4\alpha+4}{2\alpha+1},1}(\mathbf{R}_+, x^{2\alpha+1}\,dx)$ and $L^{\frac{4\alpha+4}{2\alpha+3},\infty}(\mathbf{R}_+, x^{2\alpha+1}\,dx)$. We have

$$\begin{aligned} |H_R f(x)| &= \Big| \int_0^\infty \sqrt{Ry}J_\alpha(Ry)(xy)^{-\alpha} \frac{(xy)^{-1/2} - y^{-1}}{x-y} f(y) y^{2\alpha+1}\,dy \Big| \\ &\le \|\sqrt{t}J_\alpha(t)\|_\infty \int_0^\infty \left| -\frac{x^{-\alpha-1/2} y^{-\alpha-3/2}}{1+\sqrt{x/y}} \right| |f(y)| y^{2\alpha+1}\,dy \\ &\le \|\sqrt{t}J_\alpha(t)\|_\infty x^{-\alpha-1/2} \int_0^\infty y^{-\alpha-3/2} |f(y)| y^{2\alpha+1}\,dy \\ &\le c \|\sqrt{t}J_\alpha(t)\|_\infty \|y^{-\alpha-3/2}\|_{\frac{4\alpha+4}{2\alpha+3},\infty} \|f\|_{\frac{4\alpha+4}{2\alpha+1},1} x^{-\alpha-1/2}\,. \end{aligned}$$

The maximal operator associated to the operators $\{H_R\}$ is therefore dominated by the function $x^{-\alpha-1/2}$, so that it is bounded from $L^{\frac{4\alpha+4}{2\alpha+1},1}(\mathbf{R}_+, x^{2\alpha+1}\,dx)$ into $L^{\frac{4\alpha+4}{2\alpha+1},\infty}(\mathbf{R}_+, x^{2\alpha+1}\,dx)$.

The boundedness on the Lorentz space $L^{\frac{4\alpha+4}{2\alpha+1},1}(\mathbf{R}_+, x^{2\alpha+1}\,dx)$ of the individual operators in the family $\{K_R\}$ is an immediate consequence of the boundedness of the Hilbert transform on the spaces $L^p(\mathbf{R}_+, x^{-\alpha p+2\alpha+1}dx)$, $2 < p < (2\alpha+2)/\alpha$. Therefore at this point we already have a proof of S. Chanillo's result. However, to study the maximal operator associated to the operators $\{K_R\}$ we need the following lemma, which is essentially contained in the paper of E. Prestini (1988).

LEMMA.
$$\sup_R |K_R f(x)| \le c\{M_1 f(x) + M_2 f(x) + Cf(x)\},$$
where
$$M_1 f(x) = \frac{1}{x}\int_0^x (y/x)^\alpha |f(y)|\, dy,$$
$$M_2 f(x) = \int_x^\infty (y/x)^\alpha |f(y)| \frac{dy}{y},$$
$$Cf(x) = \sup_R \Big| \int_{x/2}^{2x} (y/x)^\alpha \frac{\exp(iRy)}{x-y} f(y)\, dy \Big|.$$

Sketch of the proof. It is clear that
$$\Big| \int_0^{x/2} (y/x)^\alpha \frac{\sqrt{Ry}J_\alpha(Ry)}{x-y} f(y)\, dy \Big| \le \frac{c}{x}\int_0^x (y/x)^\alpha |f(y)|\, dy,$$
$$\Big| \int_{2x}^\infty (y/x)^\alpha \frac{\sqrt{Ry}J_\alpha(Ry)}{x-y} f(y)\, dy \Big| \le c\int_x^\infty (y/x)^\alpha |f(y)| \frac{dy}{y}.$$

Also, by the asymptotic expansion of Bessel functions
$$\sqrt{t}J_\alpha(t) = \sqrt{2/\pi}\cos(t - \alpha\pi/2 - \pi/4) + E_\alpha(t),$$
$tE_\alpha(t)$ and $t\frac{d}{dt}E_\alpha(t)$ bounded, we have
$$\Big| \int_{x/2}^{2x} (y/x)^\alpha \frac{\sqrt{Ry}J_\alpha(Ry)}{x-y} f(y)\, dy \Big|$$
$$\le \sqrt{2/\pi}\Big| \int_{x/2}^{2x} (y/x)^\alpha \frac{\cos(Ry - \alpha\pi/2 - \pi/4)}{x-y} f(y)\, dy \Big|$$
$$+ \Big| \int_{x/2}^{2x} (y/x)^\alpha \frac{E_\alpha(Ry)}{x-y} f(y)\, dy \Big|.$$

The first term is controlled by the Carleson operator $Cf(x)$, while it can be shown that the error is controlled by a combination of the maximal Hilbert transform, hence by $Cf(x)$, and of the Hardy–Littlewood maximal functions $M_1 f(x)$ and $M_2 f(x)$. See the paper of E. Prestini for the details. ∎

Now we can conclude the study of the maximal operator associated to the operators $\{K_R\}$.

By Hardy's inequalities the operator M_1 is bounded on every $L^p(\mathbf{R}_+, x^{2\alpha+1}\,dx)$ with $2 < p < \infty$, while M_2 is bounded on every $L^p(\mathbf{R}_+, x^{2\alpha+1}\,dx)$ with $1 \le p < \infty$ if $-1/2 \le \alpha \le 0$, or with $1 \le p < (2\alpha+2)/\alpha$ if $\alpha > 0$. Also, by the theory of A_p weights, the Carleson operator C is bounded on $L^p(\mathbf{R}_+, x^{2\alpha+1}\,dx)$ for $2 < p < (2\alpha+2)/\alpha$. Hence, by interpolation, these operators are also bounded on the Lorentz space $L^{\frac{4\alpha+4}{2\alpha+1},1}(\mathbf{R}_+, x^{2\alpha+1}\,dx)$, and a fortiori they are of restricted weak type.

This concludes the proof of the theorem in the case $p = \frac{4\alpha+4}{2\alpha+1}$. The proof in the case $p = \frac{4\alpha+4}{2\alpha+3}$ is similar. One has to decompose the operators $\{T_R\}$ as

$$\begin{aligned} T_R f(x) &= \int_0^\infty (xy)^{-\alpha-1/2} \frac{\sqrt{Ry}\,J_\alpha(Ry)}{x-y} f(y) y^{2\alpha+1}\,dy \\ &= \int_0^\infty \sqrt{Ry}\,J_\alpha(Ry)(xy)^{-\alpha} \frac{(xy)^{-1/2} - x^{-1}}{x-y} f(y) y^{2\alpha+1}\,dy \\ &\quad + \int_0^\infty (y/x)^{\alpha+1} \frac{\sqrt{Ry}\,J_\alpha(Ry)}{x-y} f(y)\,dy, \end{aligned}$$

and argue as before. ∎

Proof of Theorem II. Parts (i) and (ii) of this theorem are essentially due to C. Kenig and P. Tomas (1980), and to J. L. Rubio de Francia (1989), respectively. We reproduce here their arguments for the sake of completeness.

(i) By the asymptotic expansion of Bessel functions, if $1 < y \ll x$ we have

$$\begin{aligned} &(xy)^{-\alpha} \frac{Rx J_{\alpha+1}(Rx) J_\alpha(Ry) - Ry J_{\alpha+1}(Ry) J_\alpha(Rx)}{x^2 - y^2} \\ &\quad \approx \frac{2}{\pi} \cos\left(Rx - \frac{\pi(\alpha+1)}{2} - \frac{\pi}{4}\right) x^{-\alpha-3/2} \cos\left(Ry - \frac{\pi\alpha}{2} - \frac{\pi}{4}\right) y^{-\alpha-1/2}. \end{aligned}$$

Hence, if the function f is chosen appropriately in the unit ball of $L^{\frac{4\alpha+4}{2\alpha+3},r}(\mathbf{R}_+, x^{2\alpha+1}\,dx)$ and with support in the interval $[1,k]$, and if $x \gg k$, then

$$S_R f(x) \approx \|y^{-\alpha-1/2}\chi_{[1,k]}(y)\|_{\frac{4\alpha+4}{2\alpha+1},s}\, x^{-\alpha-3/2},$$

$1/r + 1/s = 1$. But if $s < \infty$, then $\|y^{-\alpha-1/2}\chi_{[1,k]}(y)\|_{\frac{4\alpha+4}{2\alpha+1},s} \to \infty$ as $k \to \infty$.

(ii) It is well known that the operators $\{S_R\}$ do not map the space of test functions into $L^{\frac{4\alpha+4}{2\alpha+3},s}(\mathbf{R}_+, x^{2\alpha+1}\,dx)$ if $s < \infty$. Hence, by duality, if $r > 1$ the operators $\{S_R\}$ cannot map $L^{\frac{4\alpha+4}{2\alpha+1},r}(\mathbf{R}_+, x^{2\alpha+1}\,dx)$ into the space of tempered distributions. ∎

We conclude with the remark that Theorem I was also obtained by A. Crespi (1989), and E. Romera and F. Soria (1989), using the techniques in Chanillo's paper. Y. Kanjin (1988) also studied the convergence and divergence of spherical means for radial functions in L^p spaces using transplantation with Jacobi expansions.

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UNIVERSITÀ DELLA CALABRIA
ARCAVACATA DI RENDE, COSENZA
ITALY

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Added in proof (October 1990). For functions on $\mathbf{R}^N$ define $Sf(x) = \int_{|\xi|\leq 1} \widehat{f}(\xi)e^{2\pi i\xi x}d\xi$. C. Fefferman (1971) proved that the operator S is bounded on $L^p(\mathbf{R}^N)$ only if $p = 2$. Indeed, it is possible to show that this operator is not bounded on the Lorentz spaces $L^{2,r}(\mathbf{R}^N)$ if $r \neq 2$.