COLLOQUIUM MATHEMATICUM

VOL. LIX 1990 FASC. 2

ON JORDAN $*$-DERIVATIONS AND AN APPLICATION*

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1. Introduction. Let $\mathscr{A}$ be a real or complex Banach $*$-algebra. An additive (linear) mapping $D: \mathscr{A} \to \mathscr{A}$ is called an *additive* (*linear*) *Jordan derivation* if $D(a^2) = aD(a)+D(a)a$ for all $a \in \mathscr{A}$. An additive (linear) mapping $D: \mathscr{A} \to \mathscr{A}$ is called an *additive* (*linear*) *Jordan $*$-derivation* if $D(a^2) = aD(a)+D(a)a^*$ for all $a \in \mathscr{A}$. It is easy to see that for an arbitrary element $b \in \mathscr{A}$ the function $D_b: \mathscr{A} \to \mathscr{A}$ defined by $D_b(a) = ab-ba$ ($D_b(a) = ab-ba^*$) is a Jordan derivation (Jordan $*$-derivation). Such mappings are called *inner Jordan derivations* (*Jordan $*$-derivations*).

Let X be a real or complex Banach space. We denote by $\mathscr{B}(X)$ the algebra of all bounded linear operators from X into itself. One can verify that $\mathscr{B}(X)$ is a prime ring, that is, $A, B \in \mathscr{B}(X)$ and $A\mathscr{B}(X)B = \{0\}$ imply $A = 0$ or $B = 0$. Using the result of Herstein [4], which states that all Jordan derivations on prime rings of characteristic different from two are derivations, we infer that all additive Jordan derivations on $\mathscr{B}(X)$ satisfy $D(AB) = AD(B) + D(A)B$ for all pairs $A, B \in \mathscr{B}(X)$. Suppose first that X is infinite dimensional. Then there are no non-zero central idempotents $T \in \mathscr{B}(X)$ having the property that $T\mathscr{B}(X)$ is of finite dimension. This fact and the result of Johnson and Sinclair [5] yield that all additive derivations on $\mathscr{B}(X)$ are linear derivations. According to a well-known theorem due to Chernoff [1] we conclude finally that all additive Jordan derivations on $\mathscr{B}(X)$ are inner. The same result for linear Jordan derivations was proved in [16]. Now we shall assume that X is finite dimensional. Then $\mathscr{B}(X)$ is the algebra of all $(n \times n)$-matrices. Using the same approach as before we see that all linear Jordan derivations on $\mathscr{B}(X)$ are inner. As the following example shows the same conclusion cannot be obtained for additive Jordan derivations. Let us first recall that an additive derivation on the field of real numbers or on the field of complex numbers vanishes at every algebraic number. On the other hand, if t is a transcendental number, then there is an additive derivation f which does not vanish at t (see [8]). It follows that a non-trivial derivation f is not continuous. A simple calculation shows that a mapping D on the algebra of all $(n \times n)$-matrices

* This work was supported by the Research Council of Slovenia.

defined by $D([a_{ij}]) = [f(a_{ij})]$, where f is a non-trivial derivation, is an additive Jordan derivation which is not continuous, and therefore D is not inner.

It is the aim of this paper to obtain similar results for additive Jordan $*$-derivations defined on $\mathscr{B}(H)$, where H is a Hilbert space. First we will consider the case where H is a complex Hilbert space. We need the following result:

THEOREM 1.1. *Let $\mathscr{A}$ be a real or complex Banach $*$-algebra with an identity element and D: $\mathscr{A} \to \mathscr{A}$ an additive function. Then the following assertions are equivalent*:

(i) *D is a Jordan $*$-derivation*;

(ii) *for all invertible $a \in \mathscr{A}$, $D(a) = -aD(a^{-1})a^*$*;

(iii) *for all pairs $a, b \in \mathscr{A}$, $D(aba) = abD(a) + aD(b)a^* + D(a)b^*a^*$.*

For the proof of this theorem we refer the reader to [11]. In [15] it has been proved that an additive function f defined on a complex Banach $*$-algebra with identity element e, which satisfies the relation $f(a) = -af(a^{-1})a^*$ for all invertible $a \in \mathscr{A}$, is of the form $2if(a) = af(ie) - f(ie)a^*$. This result can be obtained as a direct consequence of Theorem 1.1 as well. All we need to do is to replace the element a in (iii) by i. Thus all additive Jordan $*$-derivations on $\mathscr{B}(H)$, where H is a complex Hilbert space, are inner. Let us now assume that H is a real Hilbert space, $\dim H = 1$. In this special case non-zero additive Jordan $*$-derivations on $\mathscr{B}(H)$ can be considered as non-trivial additive derivations on $\boldsymbol{R}$ which are not inner. So it remains to consider the case where H is a real Hilbert space, $\dim H > 1$. In the next section we shall prove that in this case all additive Jordan $*$-derivations on $\mathscr{B}(H)$ are inner.

The study of Jordan $*$-derivations is motivated by some problems concerning representability of quadratic functionals. An application of the previous result in this theory will be treated at the end.

2. Jordan $*$-derivations on $\mathscr{B}(H)$. We begin this section by proving two lemmas which will be needed in the sequel.

LEMMA 2.1. *Let $\mathscr{A}$ be a real Banach $*$-algebra with an identity element and D: $\mathscr{A} \to \mathscr{A}$ an additive Jordan $*$-derivation. Then*

$$D(ab) + D(ba) = aD(b) + bD(a) + D(b)a^* + D(a)b^*$$

for all pairs $a, b \in \mathscr{A}$.

Proof. All we need to do is to put $a+b$ instead of a in the relation $D(a^2) = aD(a) + D(a)a^*$.

LEMMA 2.2. *Let H be a real Hilbert space*, $\dim H > 1$, *and let $\mathscr{B}(H)$ be the algebra of all bounded linear operators on H. Then each additive Jordan $*$-derivation on $\mathscr{B}(H)$ is linear.*

Proof. For an arbitrary operator $S\in\mathscr{B}(H)$ and $t\in\boldsymbol{R}$ we get, using Theorem 1.1 (iii) and Lemma 2.1,

$$tSD(S)+SD(tI)S^*+tD(S)S^* = D(S(tI)S)$$
$$= \tfrac{1}{2}D((tI)S^2)+\tfrac{1}{2}D(S^2(tI)) = tD(S^2)+\tfrac{1}{2}(S^2D(tI)+D(tI)S^{*2})$$

or, equivalently,

$$(1)\qquad 2SD(tI)S^* = S^2D(tI)+D(tI)S^{*2}.$$

In particular, for an arbitrary orthogonal projection $P\in\mathscr{B}(H)$ we have

$$2PD(tI)P = PD(tI)+D(tI)P.$$

Multiplying this relation by P first from the left side, then from the right side, and comparing the relations so obtained, we get $PD(tI) = D(tI)P$. It follows that $D(tI)$ is of the form $D(tI) = f(t)I$. Suppose that there exists a real number t_0 having the property $f(t_0)\neq 0$. Then by substituting $t = t_0$ the relation (1) goes over into $2SS^* = S^2+S^{*2}$ for all $S\in\mathscr{B}(H)$. As a consequence, all skew-hermitian operators S satisfy $S^2 = 0$. This contradiction implies $D(tI)\equiv 0$. Using Lemma 2.1 we obtain finally

$$D(tS) = \tfrac{1}{2}D((tI)S+S(tI)) = tD(S).$$

THEOREM 2.3. *Let H be a real Hilbert space*, $\dim H > 1$, *and let $\mathscr{B}(H)$ be the algebra of all bounded linear operators on H. Suppose that D: $\mathscr{B}(H)\to\mathscr{B}(H)$ is an additive Jordan *-derivation. Then there exists a unique linear operator $T\in\mathscr{B}(H)$ such that $D(S) = ST-TS^*$ holds for all $S\in\mathscr{B}(H)$.*

Proof. The uniqueness of T is obvious. In the proof of the existence we shall first consider the case where H is $2n$-dimensional, $n\in\boldsymbol{N}$, or infinite dimensional. In both cases we can find a complete orthonormal set of the form $\{e_\alpha, f_\alpha;\ \alpha\in J\}$. We define three operators $A, B, C\in\mathscr{B}(H)$ by

$$Ae_\alpha = -f_\alpha,\quad Be_\alpha = e_\alpha,\quad Ce_\alpha = f_\alpha,$$
$$Af_\alpha = e_\alpha,\quad Bf_\alpha = -f_\alpha,\quad Cf_\alpha = e_\alpha$$

for all $\alpha\in J$. The following relations are easy to verify:

$$(2)\qquad A^* = -A = A^{-1},\quad B^* = B = B^{-1},\quad C^* = C = C^{-1},$$

$$(3)\qquad AB = -C = -BA,\quad AC = B = -CA,\quad BC = A = -CB.$$

It follows from Theorem 1.1 (ii) and the relations (2) that

$$(4)\qquad AD(A) = D(A)A,\quad BD(B) = -D(B)B,\quad CD(C) = -D(C)C.$$

We recall that an operator S *anticommutes* with an operator P if and only if $SP = -PS$. Let us denote:

by $\mathscr{A}_1$ the set of all linear bounded operators on H which commute with A, B and C;

by $\mathscr{A}_2$ the set of all operators $S \in \mathscr{B}(H)$ which commute with A and anticommute with B and C;

by $\mathscr{A}_3$ the set of all operators $S \in \mathscr{B}(H)$ which commute with B and anticommute with A and C;

by $\mathscr{A}_4$ the subspace of all those operators $S \in \mathscr{B}(H)$ which commute with C and anticommute with A and B.

Each operator $S \in \mathscr{B}(H)$ can be written as

$$S = \tfrac{1}{4}\big((S - ASA + BSB + CSC) + (S - ASA - BSB - CSC)$$
$$+ (S + ASA + BSB - CSC) + (S + ASA - BSB + CSC)\big).$$

Therefore $\mathscr{B}(H)$ may be described as a direct sum

$$\mathscr{B}(H) = \mathscr{A}_1 \oplus \mathscr{A}_2 \oplus \mathscr{A}_3 \oplus \mathscr{A}_4.$$

Let us define

$$T = \tfrac{1}{2}\big(AD(C)B - AD(A) - D(B)B\big).$$

For $S \in \mathscr{A}_1$ we have $D(S) = -D(A^2S) = -D(ASA)$. Using Theorem 1.1 (iii) we get

$$D(S) = -SAD(A) - AD(S)A^* - D(A)A^*S^*.$$

Now, the relations (2) and (4) yield

$$D(S) - AD(S)A = S\big(-AD(A)\big) - \big(-AD(A)\big)S^*. \tag{5}$$

Similarly, we obtain

$$\begin{aligned} D(S) - BD(S)B &= S\big(-D(B)B\big) - \big(-D(B)B\big)S^*, \\ D(S) - CD(S)C &= S\big(-D(C)C\big) - \big(-D(C)C\big)S^*. \end{aligned} \tag{6}$$

Multiplying the last relation from both sides by A we get

$$AD(S)A + BD(S)B = S\big(AD(C)B\big) - \big(AD(C)B\big)S^*. \tag{7}$$

It follows from (5)–(7) and the definition of T that $D(S) = ST - TS^*$. Exactly as before, for an arbitrary operator $S \in \mathscr{A}_2$ we obtain the following equations:

$$D(S) - AD(S)A = S\big(-AD(A)\big) - \big(-AD(A)\big)S^*,$$
$$D(S) + BD(S)B = S\big(-D(B)B\big) - \big(-D(B)B\big)S^*,$$
$$D(S) + CD(S)C = S\big(-D(C)C\big) - \big(-D(C)C\big)S^*.$$

We multiply the last relation from both sides by A in order to obtain $D(S) = ST - TS^*$. The relation $D(S) = ST - TS^*$ is obtained for $S \in \mathscr{A}_3 \oplus \mathscr{A}_4$ in a similar way.

It remains to prove our theorem in the case where H is finite dimensional, $\dim H = 2n+1$, $n \geqslant 1$. Then $\mathscr{B}(H)$ can be considered as the algebra of all

$(2n+1)\times(2n+1)$-matrices M^{2n+1}. In the sequel we shall use the following symbols:

$$P_k = [p_{ij}] \in M^{2n+1}, \quad k \in \{1, 2, \dots, 2n+1\},$$

$$p_{ij} = \begin{cases} 0, & i \neq j, \\ 1, & i = j \text{ and } i \neq k, \\ 0, & i = j = k, \end{cases}$$

$$E^n_{kr} = [e_{ij}] \in M^n, \quad n, k, r \in N, \ k \leqslant n, r \leqslant n,$$

$$e_{ij} = \begin{cases} 0, & (i, j) \neq (k, r), \\ 1, & (i, j) = (k, r), \end{cases}$$

$$\varphi_k\colon M^{2n} \to M^{2n+1}, \quad \varphi_k([a_{ij}]) = [b_{ij}],$$

$$b_{ij} = \begin{cases} a_{ij}, & i < k \text{ and } j < k, \\ 0, & i = k \text{ or } j = k, \\ a_{i,j-1}, & i < k \text{ and } j > k, \\ a_{i-1,j}, & i > k \text{ and } j < k, \\ a_{i-1,j-1}, & i > k \text{ and } j > k, \end{cases}$$

$$\psi_k\colon M^{2n+1} \to M^{2n}, \quad \psi_k([b_{ij}]) = [a_{ij}],$$

$$a_{ij} = \begin{cases} b_{ij}, & i < k \text{ and } j < k, \\ b_{i+1,j}, & i \geqslant k \text{ and } j < k, \\ b_{i,j+1}, & i < k \text{ and } j \geqslant k, \\ b_{i+1,j+1}, & i \geqslant k \text{ and } j \geqslant k, \end{cases}$$

$$D_k\colon M^{2n} \to M^{2n}, \quad D_k(S) = \psi_k\big(P_k D(\varphi_k(S)) P_k\big).$$

Obviously, φ_k is a $*$-homomorphism. The relations

$$\psi_k(SR) = \psi_k(S)\psi_k(R), \quad \psi_k(S+R) = \psi_k(S) + \psi_k(R), \quad \psi_k(S^*) = \psi_k(S)^*$$

hold for all $S, R \in \operatorname{Im}\varphi_k$. It is easy to verify that $\psi_k\varphi_k(S) = S$ and $\varphi_k\psi_k(R) = P_k R P_k$ for all $S \in M^{2n}$ and all $R \in M^{2n+1}$. Therefore, for an arbitrary integer k, $1 \leqslant k \leqslant 2n+1$, and $S \in M^{2n}$ we have

$$\begin{aligned} D_k(S^2) &= \psi_k\big(P_k D(\varphi_k(S^2)) P_k\big) \\ &= \psi_k\big(\varphi_k(S) P_k D(\varphi_k(S)) P_k + P_k D(\varphi_k(S)) P_k \varphi_k(S^*)\big) \\ &= SD_k(S) + D_k(S) S^*. \end{aligned}$$

Since the mapping D_k is additive, there exists a matrix T_k such that $D_k(S) = ST_k - T_k S^*$ for all $S \in M^{2n}$.

Suppose now that $n \geqslant 2$ or, equivalently, $2n+1 \geqslant 5$. We define a matrix $T = [a_{ij}] \in M^{2n+1}$ by

$$a_{ij} = \big(\varphi_k(T_k)\big)_{ij}, \quad k \notin \{i, j\}.$$

First we shall prove that T is well defined, that is, the relations $1 \leqslant i, j, m, r \leqslant 2n+1$, $m \notin \{i, j\}$ and $r \notin \{i, j\}$ imply

$$(\varphi_r(T_r))_{ij} = (\varphi_m(T_m))_{ij}.$$

For this purpose let us choose an integer $k \notin \{i, j, r, m\}$, $1 \leqslant k \leqslant 2n+1$. Then there is a unique matrix $F \in M^{2n}$ such that $E_{kj}^{2n+1} = \varphi_r(F)$. One can see that

$$(\varphi_r(T_r))_{ij} = -(\varphi_r(FT_r - T_rF^*))_{ik}.$$

We have

$$\psi_r(P_rD(E_{kj}^{2n+1})P_r) = \psi_r(P_rD(\varphi_r(F))P_r) = D_r(F) = FT_r - T_rF^*.$$

This implies $\varphi_r(D_r(F)) = P_rD(E_{kj}^{2n+1})P_r$, so that

$$(\varphi_r(T_r))_{ij} = -(\varphi_r(D_r(F)))_{ik} = -(P_rD(E_{kj}^{2n+1})P_r)_{ik} = -D(E_{kj}^{2n+1})_{ik}.$$

The relation $(\varphi_m(T_m))_{ij} = -D(E_{kj}^{2n+1})_{ik}$ can be obtained exactly in the same way. Thus the matrix T is well defined.

Let us now define a mapping $J: M^{2n+1} \to M^{2n+1}$ by $J(S) = ST - TS^*$. Since J and D are linear mappings, it is enough to prove $D(E_{kj}^{2n+1}) = J(E_{kj}^{2n+1})$, $1 \leqslant k, j \leqslant 2n+1$, in order to obtain $D = J$. For a fixed pair of integers k, j ($1 \leqslant k, j \leqslant 2n+1$) let us choose an integer r such that $1 \leqslant r \leqslant 2n+1$, $r \neq k$ and $r \neq j$. For such an r there exists a unique matrix $E \in M^{2n}$ such that $\varphi_r(E) = E_{kj}^{2n+1}$. We have

$$\begin{aligned} P_rD(E_{kj}^{2n+1})P_r &= \varphi_r(\psi_r(P_rD(E_{kj}^{2n+1})P_r)) = \varphi_r(D_r(E)) \\ &= \varphi_r(ET_r - T_rE^*) = \varphi_r(E)\varphi_r(T_r) - \varphi_r(T_r)\varphi_r(E^*) \\ &= \varphi_r(E)P_rTP_r - P_rTP_r\varphi_r(E^*) \\ &= P_r(\varphi_r(E)T - T\varphi_r(E^*))P_r = P_rJ(E_{kj}^{2n+1})P_r. \end{aligned}$$

Since $2n+1 \geqslant 5$, we can find at least three different integers r having the property

$$P_rD(E_{kj}^{2n+1})P_r = P_rJ(E_{kj}^{2n+1})P_r.$$

It follows that $D(E_{kj}^{2n+1}) = J(E_{kj}^{2n+1})$. Hence D is inner.

We shall complete our proof considering the case $\dim H = 3$. We already know that there exist real numbers a, b, c, d, e, f, g, h, i, j, k, m such that for arbitrary x, y, z, $w \in \mathbf{R}$ the following relations hold:

$$(8) \qquad P_3D\left(\begin{bmatrix} x & y & 0 \\ z & w & 0 \\ 0 & 0 & 0 \end{bmatrix}\right)P_3$$

$$= \begin{bmatrix} x & y & 0 \\ z & w & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x & z & 0 \\ y & w & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$P_2 D\left(\begin{bmatrix} x & 0 & y \\ 0 & 0 & 0 \\ z & 0 & w \end{bmatrix}\right) P_2$$

$$= \begin{bmatrix} x & 0 & y \\ 0 & 0 & 0 \\ z & 0 & w \end{bmatrix} \begin{bmatrix} e & 0 & f \\ 0 & 0 & 0 \\ g & 0 & h \end{bmatrix} - \begin{bmatrix} e & 0 & f \\ 0 & 0 & 0 \\ g & 0 & h \end{bmatrix} \begin{bmatrix} x & 0 & z \\ 0 & 0 & 0 \\ y & 0 & w \end{bmatrix},$$

$$P_1 D\left(\begin{bmatrix} 0 & 0 & 0 \\ 0 & x & y \\ 0 & z & w \end{bmatrix}\right) P_1$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & x & y \\ 0 & z & w \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & i & j \\ 0 & k & m \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ 0 & i & j \\ 0 & k & m \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & x & z \\ 0 & y & w \end{bmatrix}.$$

In particular, we have

$$P_3 D\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right) P_3 = \begin{bmatrix} 0 & b & 0 \\ -c & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$P_2 D\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right) P_2 = \begin{bmatrix} 0 & 0 & f \\ 0 & 0 & 0 \\ -g & 0 & 0 \end{bmatrix},$$

$$P_1 D\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right) P_1$$

$$= P_1 D\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) P_1$$

$$= -P_1 D\left(\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) P_1 = 0.$$

Thus

$$D\left(\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix}\right)=\begin{bmatrix}0&b&f\\-c&0&0\\-g&0&0\end{bmatrix}$$

and, similarly,

$$D\left(\begin{bmatrix}0&0&0\\0&1&0\\0&0&0\end{bmatrix}\right)=\begin{bmatrix}0&-b&0\\c&0&j\\0&-k&0\end{bmatrix},$$

$$D\left(\begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix}\right)=\begin{bmatrix}0&0&-f\\0&0&-j\\g&k&0\end{bmatrix}.$$

Now it follows from (8) that

$$D\left(\begin{bmatrix}0&1&0\\-1&0&0\\0&0&0\end{bmatrix}\right)=\begin{bmatrix}c-b&a+d&x\\-a-d&c-b&y\\u&w&z\end{bmatrix},$$

where x, y, u, w, z represent entries not yet known. These entries will be determined by using the following relations. First, we have

$$D\left(\begin{bmatrix}0&1&0\\-1&0&0\\0&0&0\end{bmatrix}^2\right)$$

$$=-D\left(\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix}+\begin{bmatrix}0&0&0\\0&1&0\\0&0&0\end{bmatrix}\right)=\begin{bmatrix}0&0&-f\\0&0&-j\\g&k&0\end{bmatrix}.$$

On the other hand,

$$D\left(\begin{bmatrix}0&1&0\\-1&0&0\\0&0&0\end{bmatrix}^2\right)$$

$$=\begin{bmatrix}0&1&0\\-1&0&0\\0&0&0\end{bmatrix}D\left(\begin{bmatrix}0&1&0\\-1&0&0\\0&0&0\end{bmatrix}\right)+D\left(\begin{bmatrix}0&1&0\\-1&0&0\\0&0&0\end{bmatrix}\right)\begin{bmatrix}0&-1&0\\1&0&0\\0&0&0\end{bmatrix}$$

$$=\begin{bmatrix}0&0&y\\0&0&-x\\w&-u&0\end{bmatrix}.$$

The comparison of these two expressions gives us $y = -f$, $x = j$, $w = g$, and $u = -k$. Using Lemma 2.1 we infer from the equation

$$0 = D\left(\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) + D\left(\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right)$$

that $z = 0$. Hence

$$D\left(\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} c-b & a+d & j \\ -a-d & c-b & -f \\ -k & g & 0 \end{bmatrix}.$$

Using a similar approach we obtain

$$D\left(\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} c-b & d-a & j \\ a-d & b-c & f \\ -k & -g & 0 \end{bmatrix}.$$

The additivity of D implies finally

$$D\left(\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} c-b & d & j \\ -d & 0 & 0 \\ -k & 0 & 0 \end{bmatrix}$$

and

$$D\left(\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & -a & 0 \\ a & b-c & f \\ 0 & -g & 0 \end{bmatrix}.$$

The same method gives us

$$D\left(\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} g-f & k & h \\ -j & 0 & 0 \\ -h & 0 & 0 \end{bmatrix},$$

$$D\left(\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 & -e \\ 0 & 0 & -c \\ e & b & f-g \end{bmatrix},$$

$$D\left(\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & -f & 0 \\ g & k-j & m \\ 0 & -m & 0 \end{bmatrix},$$

$$D\left(\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 & -b \\ 0 & 0 & -i \\ c & i & j-k \end{bmatrix}.$$

The relation

$$D\left(\begin{bmatrix}0&0&0\\0&0&0\\0&1&0\end{bmatrix}\right)$$

$$=D\left(\begin{bmatrix}0&1&0\\0&0&0\\0&0&0\end{bmatrix}\begin{bmatrix}0&0&0\\0&0&0\\1&0&0\end{bmatrix}+\begin{bmatrix}0&0&0\\0&0&0\\1&0&0\end{bmatrix}\begin{bmatrix}0&1&0\\0&0&0\\0&0&0\end{bmatrix}\right)$$

and Lemma 2.1 imply $d=i$. After proving that $a=e$ and $h=m$ the straightforward computation shows us that $D(S)=ST-TS^*$ for all $S\in M^3$, where T is defined as

$$T=\begin{bmatrix}a&b&f\\c&d&j\\g&k&h\end{bmatrix}.$$

3. Quadratic and sesquilinear functionals. We shall start this section with some definitions. Let $\mathscr{A}$ be a real or complex $*$-algebra and let X be a left $\mathscr{A}$-module. A mapping $B\colon X\times X\to\mathscr{A}$ is called a *sesquilinear functional* if

$B(\cdot,\cdot)$ is additive in both arguments,

$B(ax,y)=aB(x,y)$ and $B(x,ay)=B(x,y)a^*$ for all pairs $x,y\in X$ and all $a\in\mathscr{A}$.

A mapping $Q\colon X\to\mathscr{A}$ is called a *quadratic functional* if

$Q(x+y)+Q(x-y)=2Q(x)+2Q(y)$ for all pairs $x,y\in X$,

$Q(ax)=aQ(x)a^*$ for all $x\in X$ and all $a\in\mathscr{A}$.

A routine calculation shows that each sesquilinear functional gives rise to a quadratic functional by the relation $Q(x)=B(x,x)$. The question arises whether each quadratic functional can be represented by some sesquilinear functional. More precisely, consider the following problem:

Let $\mathscr{A}$ be a real or complex $*$-algebra and let X be a left $\mathscr{A}$-module. Suppose there exists a quadratic functional $Q\colon X\to\mathscr{A}$. Does there exist a sesquilinear functional $B\colon X\times X\to\mathscr{A}$ such that $Q(x)=B(x,x)$ for all $x\in X$?

In general the answer to this question is negative. For results concerning representability of quadratic functionals by sesquilinear functionals and for counterexamples which show that there exist quadratic functionals which cannot be represented by sesquilinear functionals we refer to [2], [3], [6], [7], and [9]–[16].

In the case where $\mathscr{A}$ is a real Banach $*$-algebra with an identity element we have the following result [11]:

THEOREM 3.1. *Let $\mathscr{A}$ be a real Banach $*$-algebra with an identity element. Let us denote the space of all Jordan $*$-derivations on $\mathscr{A}$ by $\mathscr{D}$. Let us define a mapping $F\colon\mathscr{A}\to\mathscr{D}$ by $F(a)=D_a$, $D_a(x)=xa-ax^*$ for all $x\in\mathscr{A}$. Suppose that F is one-to-one and onto. Then for each quadratic functional Q defined on an* *arbitrary $\mathscr{A}$-module X there is a sesquilinear functional B*: $X \times X \to \mathscr{A}$ *such that* $Q(x) = B(x, x)$ *for all* $x \in X$.

As an immediate consequence of this result and Theorem 2.3 we have

THEOREM 3.2. *Let X be a left $\mathscr{B}(H)$-module, where H is a real Hilbert space,* $\dim H > 1$. *Then for each quadratic functional* $Q: X \to \mathscr{B}(H)$ *there is a sesquilinear functional* $B: X \times X \to \mathscr{B}(H)$ *which gives rise to Q by the relation* $Q(x) = B(x, x)$.

Acknowledgment. I would like to thank Joso Vukman for suggesting this area of research to me.

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Reçu par la Rédaction le 18.I.1989