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A PROPERTY OF A DECOMPOSITION OF WEAKLY ALMOST PERIODIC FUNCTIONS

BY

JADWIGA PORADA (WROCŁAW)

1. Introduction. Throughout this paper G will denote a group equipped with a Hausdorff topology in which left and right translations are continuous functions from G into itself. The space of all almost periodic functions on G will be denoted by AP, and the space of all weakly almost periodic functions on G by WAP. The first of these spaces is contained in the other, and both consist of bounded complex functions. Both will be considered with the norm

$$\|f\|_\infty = \sup_{t\in G} |f(t)|, \quad \text{where } f \in \mathrm{WAP}.$$

It is known that on WAP there exists a unique two-sided invariant mean μ ([1], p. 15, Corollary 1.26) and that every function $f \in$ WAP admits a unique decomposition

$$f = f_1 + f_2, \quad \text{where } f_1 \in \mathrm{AP} \text{ and } \mu(|f_2|) = 0 \tag{1}$$

([1], p. 30, Theorem 2.22). The main result of the present paper deals with the norms of the three members in (1).

2. The functional $\tilde{\mu}$. Our general assumption on G assures the existence of the mean μ on WAP, but in what follows we need also a mean for some bounded functions beyond WAP. It will now be constructed.

Put, for an arbitrary function $h \in$ WAP with $\mu(|h|) = 0$ and for an arbitrary $\varepsilon > 0$,

$$A(h, \varepsilon) = \{t \in G\colon\ |h(t)| \geqslant \varepsilon\}, \tag{2}$$

and let χ_A denote the characteristic function of a set $A \subset G$. Consider first the space Φ_0 of all linear complex combinations

$$\varphi_0(t) = \sum_{i=1}^{n} c_i \chi_{A_i}(t), \quad \text{where } A_i \subset A(h_i, \varepsilon_i) \tag{3}$$

for some functions $h_i \in$ WAP with $\mu(|h_i|) = 0$ and for some numbers $\varepsilon_i > 0$.

It is easily seen that the space Φ_0 is two-sided translation invariant. Now we prove that

(*) *for every* $\varphi_0 \in \Phi_0$, *there exists* $t \in G$ *such that* $\varphi_0(t) = 0$.

Take a $\varphi_0 \in \Phi_0$ of form (3). Clearly,

$$\bigcup_{i=1}^{n} A_i \subset \bigcup_{i=1}^{n} A(h_i, \varepsilon_i)$$

and applying (2) we get

$$\bigcup_{i=1}^{n} A(h_i, \varepsilon_i) \subset A\Big(\sum_{i=1}^{n} |h_i|, \varepsilon\Big) \quad \text{for } \varepsilon = \min_{1\leqslant i\leqslant n} \varepsilon_i.$$

Since

$$\mu\Big(\sum_{i=1}^{n} |h_i|\Big) = 0,$$

there exists a point

$$t \in G \setminus A\Big(\sum_{i=1}^{n} |h_i|, \varepsilon\Big).$$

Thus

$$t \in G \setminus \bigcup_{i=1}^{n} A_i,$$

whence, by (3), we have $\varphi_0(t) = 0$.

We define now a linear space Φ of functions $\varphi = \varphi_0 + c$, with $\varphi_0 \in \Phi_0$ and c a complex number.

Clearly, the space Φ is also two-sided translation invariant. It follows from (*) that, for every $\varphi \in \Phi$, this decomposition is unique. Thus setting $\tilde{\mu}(\varphi) = c$, we get a linear functional $\tilde{\mu}$ on Φ which has all properties of a mean, i.e.

$\tilde{\mu}(\varphi) \geqslant 0$, whenever $\varphi \in \Phi$ and $\varphi \geqslant 0$;

$\tilde{\mu}(1) = 1$;

$\tilde{\mu}({}_s\varphi) = \tilde{\mu}(\varphi_s) = \tilde{\mu}(\varphi)$, whenever $\varphi \in \Phi$ and $s \in G$.

In fact, the first property follows from (*) and the two other are obvious.

LEMMA 1. *If* $A \subset G$, $\chi_A \in \Phi$ *and* $\tilde{\mu}(\chi_A) = 0$, *and if finitely many right translations of a set* $B \subset G$ *cover* G, *i.e.*

$$\bigcup_{i=1}^{n} Bs_i = G \quad \textit{for some points } s_i \in G,$$

then $B \cap (G \setminus A) \neq \emptyset$.

Proof. Suppose, to the contrary, that $B \subset A$. It follows from the assumptions that $\chi_A \in \Phi_0$, so $\chi_B \in \Phi_0$, and hence $\tilde{\mu}(\chi_B) = 0$. Therefore, we have also $\tilde{\mu}(\chi_{Bs_i}) = 0$ for all i and, consequently,

$$\tilde{\mu}\Big(\sum_{i=1}^{n} \chi_{Bs_i}\Big) = 0,$$

by the linearity of $\tilde{\mu}$. However, the assumption on B gives

$$\sum_{i=1}^{n} \chi_{Bs_i} \geqslant 1,$$

whence we get

$$\tilde{\mu}\Big(\sum_{i=1}^{n} \chi_{Bs_i}\Big) \geqslant 1,$$

a contradiction.

3. The main result. Lemma 1 enables us to prove the following

LEMMA 2. *If $A \subset G$, $\chi_A \in \Phi$ and $\tilde{\mu}(\chi_A) = 0$, then, for every function $h \in AP$,*

$$\|h\|_\infty = \sup_{t \in G \setminus A} |h(t)|.$$

Proof. It suffices to verify that, for each $\eta > 0$, there exists a point $u \in G \setminus A$ such that $\|h\|_\infty - |h(u)| < \eta$.

Take first any point $t_0 \in G$ with

$$\|h\|_\infty - |h(t_0)| < \eta/2, \tag{4}$$

and consider the set $N_\eta = \{s \in G\colon \|h_s - h\|_\infty < \eta/2\}$.

Since $h \in$ AP by assumption, we have the equality

$$\bigcup_{i=1}^{n} N_\eta s_i = G$$

for some finite system of points $s_i \in G$ and, therefore,

$$\bigcup_{i=1}^{n} t_0 N_\eta s_i = G.$$

Hence $t_0 N_\eta \cap (G \setminus A) \neq \emptyset$ by virtue of Lemma 1. Thus there exists a point $u \in G \setminus A$ such that $u = t_0 s$ for some $s \in N_\eta$. Then $\|h_s - h\| < \eta/2$, whence

$$|h(t_0)| - |h(t_0 s)| < \eta/2. \tag{5}$$

Inequalities (4) and (5) give $\|h\|_\infty - |h(t_0 s)| < \eta$, which completes the proof.

THEOREM. *If $f \in WAP$, then, for the unique decomposition $f = f_1 + f_2$, where $f_1 \in AP$ and $\mu(|f_2|) = 0$, we have*

$$\|f_1\|_\infty \leqslant \|f\|_\infty, \tag{i}$$

$$\|f_2\|_\infty \leqslant 2\,\|f\|_\infty. \tag{ii}$$

Proof. For an arbitrary $\varepsilon > 0$ consider the set $A(f_2, \varepsilon)$ defined as in (2). For each $t \in G \setminus A(f_2, \varepsilon)$, we have $|f_2(t)| < \varepsilon$, and so

$$|f_1(t)| \leqslant |f_1(t)+f_2(t)| + |f_2(t)|;$$

therefore, $|f_1(t)| \leqslant \|f\|_\infty + \varepsilon$. Thus

$$\sup_{t \in G \setminus A(f_2, \varepsilon)} |f_1(t)| \leqslant \|f\|_\infty + \varepsilon.$$

Since $\chi_{A(f_2,\varepsilon)} \in \Phi$ and $\tilde{\mu}(\chi_{A(f_2,\varepsilon)}) = 0$, it follows by Lemma 2 that $\|f_1\|_\infty \leqslant \|f\|_\infty + \varepsilon$. Since $\varepsilon > 0$ is arbitrary, (i) follows.

To get (ii) write

$$\|f_2\|_\infty = \|f_2 + f_1 - f_1\|_\infty \leqslant \|f\|_\infty + \|f_1\|_\infty \leqslant 2\|f\|_\infty.$$

4. Remarks. I. The uniqueness of decomposition (1), i.e. $f = f_1 + f_2$ for $f \in$ WAP, permits to define the following norm on WAP:

$$\|f\| = \|f_1\|_\infty + \|f_2\|_\infty.$$

From (i) and (ii) we conclude that the norms $\|f\|$ and $\|f\|_\infty$ are equivalent; moreover, we get the estimation

$$\text{(iii)} \qquad \|f\| \leqslant 3\|f\|_\infty.$$

II. If G is locally compact non-compact, then inequalities (i) and (ii), and so the number 3 in (iii), are best possible.

Namely, let $h \not\equiv 0$ be a continuous function on G with a compact support and let $g(t) = |h(t)|$. Then

$$f_2(t) = -2\frac{g(t)}{\|g\|_\infty}$$

is a continuous function with a compact support; thus $f_2 \in$ WAP and $\mu(|f_2|) = 0$ ([1], p. 41, Corollary 3.7). Taking $f_1 \equiv 1$ and $f = f_1 + f_2$, we get equalities in (i)-(iii).

III. In [2] it is stated that a WAP-function admits the unique decomposition (1) provided that G is an Abelian locally compact group. The method of proof in this case also yields inequality (i) (cf. [3]). Possibly, it can be extended to non-commutative case as well. However, the method used here is more elementary and the proof would not become easier even by specializing G in any way.

REFERENCES

[1] R. B. Burckel, *Weakly almost periodic functions on semigroups*, New York 1970.

[2] W. F. Eberlein, *The point spectrum of weakly almost periodic functions*, The Michigan Mathematical Journal 3 (1955-1956), p. 137-139.

[3] B. B. Wells, *Restrictions of Fourier transforms of continuous measures*, Proceedings of the American Mathematical Society 38 (1973), p. 92-94.

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