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POINTWISE CONVERGENCE OF FOURIER SERIES ON COMPACT LIE GROUPS

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Let G be a compact, simply connected, simple Lie group of dimension N and rank R. To every integrable function on G it is possible to associate its Fourier series

$$f(x) \approx \sum_{\lambda} d_\lambda \chi_\lambda * f(x),$$

where d_λ and χ_λ are the dimension and the character of the irreducible unitary representation λ respectively. Write

$$S_n f(x) = \sum_{\lambda \in nP} d_\lambda \chi_\lambda * f(x),$$

where nP is the dilation of a Weyl invariant convex polyhedron P.

R. J. Stanton and P. Tomas have shown that the polyhedral partial sums of Fourier series of functions in $L^p(G)$ may diverge almost everywhere (at least for $p < 2$), but they converge at almost every point when we consider central functions in $L^p(G)$, $p > 2N/(N+R)$ (see [ST]).

On the other hand, it is known that there exist central functions in $L^{2N/(N+R)}(G)$ with unbounded sequences of Fourier coefficients (see [GST] and [GT2]). This of course suggests that the polyhedral partial sums of central functions in $L^{2N/(N+R)}(G)$ may diverge.

The main result in this paper shows that at the critical index $p = 2N/(N+R)$ it is possible to obtain positive results for central functions in the Lorentz space $L^{2N/(N+R),1}(G)$.

THEOREM. *The polyhedral partial sums of central functions in the Lorentz space $L^{2N/(N+R),1}(G)$ converge almost everywhere.*

It may be interesting to notice that the index $2N/(N+R)$ is smaller than the critical index for the mean summability (cf. [CGT]).

We shall also briefly consider the problem of divergence of Fourier series, and prove that, for groups of rank 2, there exist central functions in

$L^{2N/(N+2),\infty}(G)$ whose polyhedral partial sums diverge on sets of positive measure.

Notation. Let G be an N-dimensional compact connected simple Lie group with rank R. Let $\mathbf{T}$ be a maximal torus of G, and let $\mathfrak{t}$ and $\mathfrak{g}$ be the Lie algebras of $\mathbf{T}$ and G respectively. We choose a positive system Φ in the set of the roots of G, and let $\{\alpha_1,\ldots,\alpha_R\}$ be the associated system of simple roots. We denote by W the Weyl group generated by the reflections σ_j in the hyperplanes $\alpha_j(H)=0$ $(j=1,\ldots,R)$, and we consider W acting both on $\mathfrak{t}$ and on the dual $\mathfrak{t}^*$. The Killing form B defines a positive definite inner product $(\cdot,\cdot)=-B(\cdot,\cdot)$ in $\mathfrak{t}$. For every $\lambda\in i\mathfrak{t}^*$ there exists a unique $H_\lambda\in\mathfrak{t}$ such that $\lambda(H)=i(H_\lambda,H)$ for every $H\in\mathfrak{t}$.

The vectors $H_j=4\pi i H_{\alpha_j}/\alpha_j(H_{\alpha_j})$ generate the lattice Ker(exp). The weights of the representations of G are the elements of the set $\Lambda=\{\lambda\in i\mathfrak{t}^*:\lambda(H)\in 2\pi i\mathbf{Z},\forall H\in \mathrm{Ker(exp)}\}$, and the fundamental weights are defined by the relations $\lambda_j(H_k)=2\pi i\delta_{jk}$, $j,k=1,\ldots,R$.

The set $\Sigma=\{\lambda\in\Lambda:\ \lambda=\sum_{j=1}^R m_j\lambda_j,\ m_j\in\mathbf{N}\}$ of dominant weights can be naturally identified with the set of the equivalence classes of unitary irreducible representations of G. A dominant weight λ is non-singular if $m_j>0$ for every $j=1,\ldots,R$. Moreover, if ξ is a character of $\mathbf{T}$, there exists a unique $\lambda\in i\mathfrak{t}^*$ such that

$$\xi\circ\exp H=e^{\lambda(H)}=e^{i(H_\lambda,H)},\quad H\in\mathfrak{t}.$$

For the character χ_λ and the dimension d_λ of the representation corresponding to the dominant weight λ we have the Weyl formulas:

$$\chi_\lambda(\exp H)=(\Delta(\exp H))^{-1}\sum_{\sigma\in W}\det(\sigma)e^{\sigma(\lambda+\beta)(H)},\quad H\in\mathfrak{t},$$

$$d_\lambda=\prod_{\alpha\in\Phi}(\lambda+\beta,\alpha)/(\beta,\alpha),$$

where

$$\beta=\frac{1}{2}\sum_{\alpha\in\Phi}\alpha,$$

$$\Delta(\exp H)=\sum_{\sigma\in W}\det(\sigma)e^{\sigma(\beta)(H)}=(-2i)^{|\Phi|}\prod_{\alpha\in\Phi}\sin(i\alpha(H)/2)$$

($|\Phi|$ denotes the cardinality of Φ).

Let ω be a non-singular dominant weight, and let $P(\omega)$ be the convex hull of the set $\{\sigma\omega\}_{\sigma\in W}$. As r ranges over $[1,+\infty)$, $rP(\omega)\cap\Sigma$ generates a countable family $\{P_n\}$ of distinct sets. The nth partial sums of the Fourier series of an integrable function f are defined by

$$S_n f(x) = \sum_{\lambda \in P_n} d_\lambda \chi_\lambda * f(x).$$

Proofs. For any integrable function g on $\mathbf{T}$ we define

$$\tilde{S}_n g(t) = \sum_{\lambda \in P_n} \sum_{\sigma \in W} \left(\int_{\mathbf{T}} g(s) e^{-\sigma(\lambda+\beta)(\log s)}\, ds \right) e^{\sigma(\lambda+\beta)(\log t)}.$$

The following lemma relates S_n to the abelian operator $\tilde{S}_n$.

LEMMA 1. $S_n f(x) = \Delta^{-1}(x) \tilde{S}_n(\Delta f)(x)$.

Proof. This is a direct consequence of Weyl's character formula (see e.g. [B] or [ST]). ■

LEMMA 2. *There exists a finite family of hyperplanes $\{\ell_j\}$ such that, given $\varepsilon > 0$, the kernels $\{\tilde{S}_n(x)\}$ associated to the operators $\{\tilde{S}_n\}$ are uniformly bounded outside the set $A_\varepsilon = \bigcup_j \{\ell_j + B(0,\varepsilon)\}$.*

Proof. It has been shown in [GT1] that the functions $\{\Delta(x)\tilde{S}_n(x)\}$ are uniformly bounded. Hence the lemma follows from the definition of Δ. ■

In order to make the paper self-contained, in the Appendix we shall sketch a different and more direct proof of this lemma.

In the sequel it will be enough to deal with central functions with support in a fixed neighbourhood of the origin which excludes the "antipodal" point.

LEMMA 3. *Let f be in $L^{2N/(N+R),1}(G)$, and write $\Delta f = b + g$, with $b = \Delta f \chi_{B(0,\varepsilon)}$. Then:*

(i) $b \in L^1(\mathbf{T})$;
(ii) $g \in L^r(\mathbf{T})$ *for some* $r > 1$.

Proof. It has been proved in [GT2] that Δ^{-1} is in $L^{2N/(N-R),\infty}(G)$. Hence, by Hölder's inequality for Lorentz spaces,

$$\begin{aligned} \int_{\mathbf{T}} |f(x)\Delta(x)|\, dx &= \int_G |f(g)||\Delta^{-1}(g)|\, dg \\ &\le \|f\|_{2N/(N+R),1} \|\Delta^{-1}\|_{2N/(N-R),\infty}, \end{aligned}$$

and (i) follows.

The proof of (ii) is similar: notice that, away from the origin and the antipodal point, Δ^{-1} belongs to $L^s(G)$ for some s larger than $2N/(N-R)$. See also [GR]. ■

Proof of the Theorem. Let f be in $L^{2N/(N+R),1}(G)$. Then, keeping the notation of the previous lemmas,

$$S_n f(x) = \Delta^{-1}(x)\tilde{S}_n b(x) + \Delta^{-1}(x)\tilde{S}_n g(x).$$

Since g is in $L^r(\mathrm{T})$ for some $r > 1$, $\tilde{S}_n g(x)$ converges almost everywhere. Now, b is supported in $B(0,\varepsilon)$, and we would like to conclude that $\tilde{S}_n b(x)$ converges to zero outside the support of b. This is not necessarily true, since Riemann's Localization Principle fails for several variables. However, the proof of this Principle and Lemma 2 show that $\tilde{S}_n b(x)$ is uniformly bounded, and hence it converges to zero, outside $A_{2\varepsilon}$. Since the measure of $A_{2\varepsilon}$ can be made arbitrarily small with ε, the Theorem follows. ■

We are not able to prove that there exist central functions in $L^{2N/(N+R)}(G)$ with polyhedral partial sums diverging on sets with positive measure. However, for groups of rank 2, we shall construct a central function in $L^{2N/(N+R),\infty}(G)$ with this property.

Let $\rho \in C^\infty(\mathbf{R}^2 - \{0\})$ be homogeneous of degree zero, identically 1 in a small open cone around $\mathbf{R}_+\omega$ and identically 0 outside the double of this cone, and let $\varphi(x) \approx \sum_\lambda \rho(\lambda)\chi_\lambda(x)$. We claim that φ is in $L^{2N/(N+R),\infty}(G)$. Indeed, the Dirac measure at 0 has Fourier expansion $\delta(x) \approx \sum_\lambda d_\lambda \chi_\lambda(x)$, and its fractional integral of order $|\Phi|$, $I^{|\Phi|}\delta(x) \approx \sum_\lambda |\lambda+\beta|^{-|\Phi|} d_\lambda \chi_\lambda(x)$, is in the Lorentz space $L^{2N/(N+R),\infty}(G)$. Clearly φ can be obtained by $I^{|\Phi|}\delta$ via a smooth bounded multiplier. (See e.g. [CW], [T] and [CT].)

Let $n = n(k)$ be such that the dominant vertex of P_n is $k\omega$, k integer. We shall sketch a proof that $(\tilde{S}_n - \tilde{S}_{n-1})(\Delta\varphi)(x)$ does not converge to zero on some set with positive measure. Hence $S_n\varphi$ cannot converge almost everywhere.

$(\tilde{S}_n - \tilde{S}_{n-1})(\Delta\varphi)$ is the sum of a trigonometric polynomial with spectrum in the fundamental Weyl chamber and its reflections. Indeed,

$$\begin{aligned}&(\tilde{S}_n - \tilde{S}_{n-1})(\Delta\varphi)(\exp H)\\ &= \sum_{\sigma\in W} \det(\sigma) e^{(k\omega+\beta)(\sigma H)} \Big(\rho(k\omega+\beta) + \sum_{j=1}^{2}\sum_{h>0} \rho(k\omega+\beta-h\alpha_j) e^{-h\alpha_j(\sigma H)}\Big).\end{aligned}$$

(Of course we suppose $\rho(\mu) = 0$ if μ is not a dominant weight.)

We shall prove that the main contribution to $(\tilde{S}_n - \tilde{S}_{n-1})(\Delta\varphi)(\exp H)$ is

$$\frac{1}{2i}\sum_{\sigma\in W}\sum_{j=1}^{2} \det(\sigma) e^{(k\omega+\beta)(\sigma H)} \frac{e^{\alpha_j(\sigma H)/2}}{\sin(i\alpha_j(\sigma H)/2)},$$

so that $(\tilde{S}_n - \tilde{S}_{n-1})(\Delta\varphi)$ does not converge to zero on some set with positive measure.

For fixed j and σ write $\alpha_j(\sigma H) = -2\pi i t$, $\rho(k\omega + \beta - h\alpha_j) = \psi(h)$. One easily checks that $\psi(s) = 1$ if $0 \le s \le \varepsilon n$, $\psi(s) \doteq 0$ if $s \ge \eta n$, $|\psi(s)| \le c$, and $\int_0^\infty |(d/ds)\psi(s)|\, ds = O(1)$, $\int_0^\infty |(d^2/ds^2)\psi(s)|\, ds = O(n^{-1})$. We need to estimate $\sum_{h\ge 0} \psi(h)\exp(2\pi i h t)$.

LEMMA 4. *Let ψ satisfy $\psi(s) = 1$ if $0 \le s \le \varepsilon n$, $\psi(s) = 0$ if $s \ge \eta n$, and $\int_0^\infty |(d/ds)\psi(s)|\, ds = O(1)$, $\int_0^\infty |(d^2/ds^2)\psi(s)|\, ds = O(n^{-1})$. Then for $0 < t < 1$,*

$$\sum_{k\ge 0} \psi(k)\exp(2\pi i k t) = \frac{ie^{-i\pi t}}{2\sin(\pi t)} + \frac{O(|nt|^{-1})}{\sin(\pi t)} + O(1).$$

Proof. Write

$$\begin{aligned}
&\int_0^\infty \psi(s)e^{2\pi i s t}\, ds \\
&= \sum_{k\ge 0} \psi(k)e^{2\pi i k t} \int_k^{k+1} e^{2\pi i(s-k)t}\, ds + \sum_{k\ge 0} \int_k^{k+1} (\psi(s) - \psi(k))e^{2\pi i s t}\, ds \\
&= \frac{e^{i\pi t}\sin(\pi t)}{\pi t} \sum_{k\ge 0} \psi(k)\exp(2\pi i k t) \\
&\quad + \sum_{k\ge 0} \int_0^1 \left(\int_0^s \frac{d}{du}\psi(k+u)\, du \right) e^{2\pi i(s+k)t}\, ds.
\end{aligned}$$

Hence

$$\begin{aligned}
\sum_{k\ge 0} \psi(k)\exp(2\pi i k t) &= \frac{\pi t e^{-i\pi t}}{\sin(\pi t)} \int_0^\infty \psi(s)e^{2\pi i s t}\, ds \\
&\quad - \frac{\pi t e^{-i\pi t}}{\sin(\pi t)} \sum_{k\ge 0} \int_0^1 \left(\int_0^s \frac{d}{du}\psi(k+u)\, du \right) e^{2\pi i(k+s)t}\, ds.
\end{aligned}$$

Since $\psi(0) = 1$ and $(d/ds)\psi(0) = 0$, by a repeated integration by parts we have

$$\begin{aligned}
\int_0^\infty \psi(s)e^{2\pi i s t}\, ds &= \frac{i}{2\pi t} - \frac{1}{4\pi^2 t^2} \int_0^\infty \frac{d^2}{ds^2}\psi(s)e^{2\pi i s t}\, ds \\
&= \frac{i}{2\pi t} + \frac{O(|nt|^{-1})}{t},
\end{aligned}$$

and also

$$\left| \sum_{k\ge 0} \int_0^1 \left(\int_0^s \frac{d}{du}\psi(k+u)\, du \right) e^{2\pi i(k+s)t}\, ds \right| \le \int_0^\infty \left| \frac{d}{ds}\psi(s) \right| ds = O(1),$$

and the lemma follows. ∎

Appendix. At least for groups of rank 2 we wish to produce a direct proof of Lemma 2.

Let μ_n be the dominant vertex of P_n. We have to study the "singularities" of the kernel

$$\tilde{S}_n(H) = \sum_{\lambda \in P(\mu_n+\beta)\cap\Lambda} e^{\lambda(H)}$$

for H in a fundamental domain.

LEMMA 2 bis. *$\tilde{S}_n$ is uniformly bounded with respect to n away from the hyperplanes $\alpha(H) = 0$, $\alpha \in \Phi$.*

Proof. As in Lemma 4, the idea is to substitute an integral for the sum in the definition of $\tilde{S}_n$.

Let $Q_{rs} = Q_{00} + r\lambda_1 + s\lambda_2$ where $r, s \in \mathbf{Z}$, and let $Q_{00} \subset it^*$ be the parallelogram with vertices 0, λ_1, λ_2, $\beta = \lambda_1 + \lambda_2$. Also let

$$Q_n = \bigcup_{\{r,s:\, r\lambda_1+s\lambda_2 \in P(\mu_n+\beta)\}} Q_{rs}.$$

Then

$$\begin{aligned}\int_{Q_n} e^{\mu(H)}\, d\mu &= \sum_{\lambda\in P(\mu_n+\beta)\cap\Lambda} e^{\lambda(H)} \int_{Q_{00}} e^{\mu(H)}\, d\mu \\ &= C\frac{\sin(i\lambda_1(H)/2)\sin(i\lambda_2(H)/2)}{\lambda_1(H)\lambda_2(H)} e^{\beta(H)/2} \sum_{\lambda\in P(\mu_n+\beta)\cap\Lambda} e^{\lambda(H)}.\end{aligned}$$

Hence

$$\sum_{\lambda\in P(\mu_n+\beta)\cap\Lambda} e^{\lambda(H)} = C\left(\frac{\sin(i\lambda_1(H)/2)\sin(i\lambda_2(H)/2)}{\lambda_1(H)\lambda_2(H)} e^{\beta(H)/2}\right)^{-1} \int_{Q_n} e^{\mu(H)}\, d\mu.$$

Let k be the smallest integer such that $Q_n \subseteq kP(\omega)$. Then

$$\int_{Q_n} e^{\mu(H)}\, d\mu = \int_{kP(\omega)} e^{\mu(H)}\, d\mu - \int_{kP(\omega)-Q_n} e^{\mu(H)}\, d\mu.$$

Since $it^* \cong \mathbf{R}^2$, we order anticlockwise the roots of G to get a finite sequence $\{\alpha^{(r)}\}_{r=1,\dots,|W|}$. The induced order of the vertices of $P(\omega)$ is defined by

$$\omega_{r+1} = \omega_r - 2\frac{(\omega_r, \alpha^{(r)})}{(\alpha^{(r)}, \alpha^{(r)})}\alpha^{(r)}.$$

The difference $kP(\omega) - Q_n$ can be decomposed as the union of a fixed number of disjoint sets $\{B_j\}$, where

$$B_j = \bigcup_{s=0}^{c(k)} \{A_j - s\alpha^{(j)}\},$$

with $c(k) \approx k$ as $k \to \infty$, and the A_j's are suitable subsets of it^* (which do not depend on k). Therefore

$$\int_{B_j} e^{\mu(H)}\, d\mu = \sum_{s=0}^{c(k)} e^{s\alpha^{(j)}(H)} \int_{A_j} e^{\mu(H)}\, d\mu$$

and

$$\int_{kP(\omega)-Q_n} e^{\mu(H)}\, d\mu = \sum_j c_j \frac{\sin(\frac{1}{2}i(c(k)+1)\alpha^{(j)}(H))}{\sin(i\alpha^{(j)}(H))}.$$

If $T_r \subset it^*$ is the triangle of vertices 0, $k\omega_r$, $k\omega_{r+1}$, we have

$$\begin{aligned}\int_{T_r} e^{\lambda(H)}\, d\lambda &= \det(\omega_r, \omega_{r+1}) \int_0^k e^{t\omega_{r+1}(H)} \int_0^{k-t} e^{s\omega_r(H)}\, ds\, dt \\ &= \det(\omega_r, \omega_{r+1}) \left(\frac{e^{k\omega_{r+1}(H)} - 1}{\omega_{r+1}(H)(\omega_{r+1} - \omega_r)(H)} - \frac{e^{k\omega_r(H)} - 1}{\omega_r(H)(\omega_{r+1} - \omega_r)(H)} \right)\end{aligned}$$

where we let $\det(\lambda, \mu) = a_1 b_2 - a_2 b_1$ if $\lambda = a_1\lambda_1 + a_2\lambda_2$ and $\mu = b_1\lambda_1 + b_2\lambda_2$. Hence

$$\begin{aligned}\int_{kP(\omega)} e^{\lambda(H)}\, d\lambda &= \sum_{r=1}^{|W|} \int_{T_r} e^{\lambda(H)}\, d\lambda \\ &= \sum_{r=1}^{|W|} \left(e^{k\omega_r(H)} - 1\right) \frac{\det(\omega_r - \omega_{r-1}, \omega_{r+1} - \omega_r)}{(\omega_r - \omega_{r-1})(H)(\omega_{r+1} - \omega_r)(H)} \\ &= \sum_{r=1}^{|W|} e^{k\omega_r(H)} \frac{\det(\omega_r - \omega_{r-1}, \omega_{r+1} - \omega_r)}{(\omega_r - \omega_{r-1})(H)(\omega_{r+1} - \omega_r)(H)}.\end{aligned}$$

Of course $\omega_{|W|+1} = \omega_1$ and $\omega_0 = \omega_{|W|}$; moreover we have used the equality

$$\frac{\det(\lambda, \mu)}{\lambda(H)\mu(H)} + \frac{\det(\mu, \nu)}{\mu(H)\nu(H)} = \frac{\det(\lambda, \nu)}{\lambda(H)\nu(H)}.$$

The above estimates together imply the lemma. ∎

We remark that the proof of Lemma 2 bis holds also for more general polyhedra (for example when the weight ω is singular).

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