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ABSOLUTE CONVERGENCE OF FOURIER SERIES ON FINITE-DIMENSIONAL GROUPS

BY

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1. Introduction. In 1914 Bernstein [1] announced the result that functions in $\mathrm{Lip}(\alpha)$ with $\alpha > 1/2$ have absolutely convergent Fourier series, while for $\alpha < 1/2$ there are functions in $\mathrm{Lip}(\alpha)$ whose Fourier series do not converge absolutely. This was generalized by Szász [9], [10] who proved that if $f \in \mathrm{Lip}(\alpha; p)$, then $\hat{f} \in l^r$, where $\alpha > 1/p + 1/r - 1$ if $1 < p \leqslant 2$, and $\alpha > 1/r - 1/2$ if $p > 2$. Szász also gave examples to show that the range of values of α could not be extended. Here the generalized Lipschitz space $\mathrm{Lip}(\alpha; p)$ is defined by

$$\mathrm{Lip}(\alpha; p) = \{f \in L^p(\boldsymbol{T}) \colon \|\tau_a f - f\|_p = O(|a|^\alpha),\ a \to 0\},$$

where $\tau_a f \colon x \to f(x-a)$ and $\boldsymbol{T}$ denotes the circle group; the notation $\mathrm{Lip}(\alpha)$ is standard for $\mathrm{Lip}(\alpha; \infty)$, in which case the functions are taken to be continuous.

Subsequently, the result was extended to other groups: by Titchmarsh [11] to the real line ($1 < p \leqslant 2$), by Fine [5] to the Cantor group ($p = \infty$), by Vilenkin [12] to compact metric abelian groups with primary character groups ($p = \infty$), by Walker [13], [14] to finite-dimensional compact metric abelian groups ($p = \infty$), and by Onneweer [7], [8] to the so-called bounded Vilenkin groups ($1 \leqslant p \leqslant \infty$). Titchmarsh and Onneweer gave examples to show that their results were best possible.

Here we show that the statement of Bernstein's theorem (and its extension by Szász) is valid for a wide class of metrizable locally compact abelian groups, including finite-dimensional groups, the proof following from a generalization of Jackson's theorem (see [3] and [4]). This will be the main result in Section 2. It will also be shown that the result is sharp for the compact solenoidal groups $\Sigma_{\boldsymbol{a}}$. For other groups the question remains open.

Throughout G will denote an infinite locally compact metric abelian group with translation invariant metric d and character group Γ_G. We shall choose Haar measures λ, θ for G, Γ_G, respectively, so that Plancherel's theorem is valid, with λ normalized in the usual way when G is compact (see [6], (31.1)). The real line $\boldsymbol{R}$ will be taken with its usual Euclidean metric, as will the circle group $\boldsymbol{T} = \boldsymbol{R}/\boldsymbol{Z}$ ($\boldsymbol{Z}$ is the group of integers). For the group $\Delta_{\boldsymbol{a}}$ of $\boldsymbol{a}$-adic integers, $\boldsymbol{a} = (a_0, a_1, \ldots)$, $a_i \geqslant 2$, consider the basis (Λ_n) of open neighbourhoods of the identity, given by

$$\Lambda_n = \{v \in \Delta_{\boldsymbol{a}} : v_k = 0 \text{ for } k < n\}.$$

We give $\Delta_{\boldsymbol{a}}$ the metric d defined by

$$d(x, y) = \begin{cases} \beta_{n+1}, & x-y \in \Lambda_n \setminus \Lambda_{n+1}, \\ \beta_1, & x-y \notin \Lambda_1, \\ 0, & x = y, \end{cases}$$

where $\beta_n = \lambda(\Lambda_n)$. It is easily verified that d is a translation invariant metric on G compatible with the given topology.

The compact solenoid $\Sigma_{\boldsymbol{a}}$ is defined as $(\boldsymbol{R} \times \Delta_{\boldsymbol{a}})/B$, where B is the infinite cyclic subgroup of $\boldsymbol{R} \times \Delta_{\boldsymbol{a}}$ generated by $(1, \boldsymbol{u})$, $\boldsymbol{u} = (1, 0, 0, \ldots)$. A (translation invariant) metric on $\Sigma_{\boldsymbol{a}}$ will be specified by

$$d\big((x, v)+B, B\big) = \inf\big\{\max\{|y|, d(w, 0)\}: (y, w) \in (x, v)+B\big\};$$

this is just the metric assigned in the usual way to quotients and finite products. The character group of $\Sigma_{\boldsymbol{a}}$ can be identified with a subgroup of the group $\boldsymbol{Q}$ of rational numbers, where $\chi_{n,l} \in \Gamma_{\Sigma_{\boldsymbol{a}}}$ corresponds to $l/a_0 a_1 \ldots a_n \in \boldsymbol{Q}$ via

$$\chi_{n,l}\big((x, v)+B\big) = \exp\left[2\pi i \frac{l}{a_0 a_1 \ldots a_n}\big(x-(v_0+a_0 v_1+ \ldots +a_0 a_1 \ldots a_{n-1} v_n)\big)\right]$$

(for details of the $\boldsymbol{a}$-adic integers, $\Sigma_{\boldsymbol{a}}$, and their character groups see [6], Sections 10, 25).

Now consider a finite-dimensional group G of topological dimension m. Using the structure theorem ([6], Theorem (24.30)) for locally compact abelian groups $G \cong \boldsymbol{R}^{m-k} \times G_0$, where $k \leqslant m$ and G_0 has a compact open subgroup K of dimension k. Furthermore, appealing to [14], Lemma 1, $K \cong (\Delta^\infty \times \Sigma^k)/H$, where Δ, Σ are the groups $\Delta_{\boldsymbol{a}}$, $\Sigma_{\boldsymbol{a}}$ with $\boldsymbol{a} = (2, 3, \ldots)$, Δ^∞ denotes the countable direct product of the groups Δ, and H is a closed 0-dimensional subgroup of $\Delta^\infty \times \Sigma^k$. A metric d will be defined for G as follows. Take a metric for Δ^∞ as for Δ, with (Λ_n) replaced by (V'_n), where

$$V'_n = \{v \in \Delta^\infty : v_{ik} = 0,\, i, k < n\},\, v = (v^{(0)}, v^{(1)}, \ldots) \text{ and } v^{(i)} = (v_{i0}, v_{i1}, \ldots\,),$$

a metric for Δ (in the definition of Σ) with respect to (Λ_n), and in both cases put $\beta_n = 2^{-n^3}$. This gives a metric d' for K, which can be extended to G_0 by

$$d''(x,y) = \begin{cases} d'(x,y), & x-y \in K, \\ 1 & \text{otherwise,} \end{cases}$$

and then to G in the usual way.

Finally, the characteristic function of a set E will be denoted by ξ_E, and wherever C appears it denotes a positive constant, not necessarily the same from line to line.

2. Bernstein's theorem. We shall say that G has *property* $P(V_n, k_n, T_n)$ if there are a basis (V_n) of symmetric open neighbourhoods of zero and corresponding families (k_n) and (T_n) of nonnegative continuous functions and compact subsets of Γ_G, respectively, such that, for each positive integer n, $\operatorname{supp}(k_n) \subset \mathscr{V}_n$ (the open subgroup of G generated by V_n), $\hat{k}_n(0) = 1$, $\operatorname{supp}(\hat{k}_n) \subset T_n$, and

$$\int_{\mathscr{V}_n} m_{V_n} k_n \, d\lambda \leqslant C,$$

where m_{V_n} is the integer-valued function on $\mathscr{V}_n$ defined by

$$m_{V_n}(x) = \min\{m \in \mathbf{Z}^+ : x \in mV_n\}.$$

With this definition we have the following analogue of Jackson's theorem (for a proof, see [4], Theorem 3).

THEOREM 1. *Assume G satisfies property $P(V_n, k_n, T_n)$ and let*

$$\omega(p; f; V_n) = \sup\{\|\tau_a f - f\|_p : a \in V_n\}.$$

Then

$$\|k_n * f - f\|_p \leqslant C\omega(p; f; V_n)$$

for every $f \in L^p(G)$ if $p \in [1, \infty)$ or for every bounded uniformly continuous f if $p = \infty$.

It was shown in [3] that all metrizable locally compact abelian groups satisfy property $P(V_n, k_n, T_n)$ for suitable families (V_n), (k_n), and (T_n). However, in many applications in approximation theory (for example, Theorem 2 below) it is important that the T_n do not increase too quickly as the V_n decrease. When $G = \mathbf{R}, \mathbf{T}$ or is 0-dimensional, or G is a group formed from these by taking quotients and finite products, families (V_n), (k_n), and (T_n) can be computed explicitly (see [4]), and the relation between V_n and T_n seems to be optimal.

From Theorem 1 we can deduce properties of the Fourier transforms of functions in $\mathrm{Lip}(\alpha; p)$ (defined as in Section 1 with $|a|$ replaced by $d(a, 0)$; when $p = \infty$, the functions are taken to be continuous). A growth condition needs to be placed on the families (V_n), (T_n), here given in terms of the radius $\delta(V_n) = \sup\{d(a, 0) : a \in V_n\}$ of V_n and the Haar measure of T_n.

THEOREM 2. *Suppose G satisfies property $P(V_n, k_n, T_n)$, where*

$$\sum_{n=1}^{\infty} \delta(V_n)^{\varepsilon}\, \theta(T_{n+1} \setminus T_n)^{\varepsilon'} < \infty \quad \textit{for } \varepsilon > \varepsilon'.$$

Then, for $1 \leqslant p \leqslant 2$, $\mathrm{Lip}(a;p)^{\wedge} \subset L^r(\Gamma_G)$, where $a > 1/p + 1/r - 1 \geqslant 0$ (with the convention that $\infty^{-1} = 0$).

If, in addition, G is compact, then $\mathrm{Lip}(a;p)^{\wedge} \subset l^r(\Gamma_G)$ for $p > 2$, where $a > 1/r - 1/2$.

Proof. First consider $1 \leqslant p \leqslant 2$. We can use Theorem 1 and the Hausdorff-Young inequality to obtain for $f \in \mathrm{Lip}(a;p)$

$$\|\hat{k}_n \hat{f} - \hat{f}\|_{p'} \leqslant C\delta(V_n)^a,$$

where p' denotes the index conjugate to p (that is, $p' = p/(p-1)$ for $p \neq 1$, and $1' = \infty$). Provided $p > 1$ we can use the fact that $\mathrm{supp}(\hat{k}_n) \subset T_n$ to write

$$\int_{T_{n+1} \setminus T_n} |\hat{f}|^{p'} d\theta \leqslant C\delta(V_n)^{ap'}$$

and, by Hölder's inequality (using the assumption that $1/p + 1/r - 1 \geqslant 0$, that is $r \leqslant p'$),

$$\int_{T_{n+1} \setminus T_n} |\hat{f}|^r d\theta \leqslant C\delta(V_n)^{ar} \Big(\int_{T_{n+1} \setminus T_n} d\theta \Big)^{1 - r/p'} = C\delta(V_n)^{ar}\, \theta(T_{n+1} \setminus T_n)^{1 - r/p'}.$$

This inequality continues to hold when $p = 1$ and $r < \infty$. Hence

$$\begin{aligned} \int_{\Gamma_G} |\hat{f}|^r d\theta &= \sum_{n=1}^{\infty} \int_{T_{n+1} \setminus T_n} |\hat{f}|^r d\theta + \int_{T_1} |\hat{f}|^r d\theta \\ &\leqslant C \sum_{n=1}^{\infty} \delta(V_n)^{ar}\, \theta(T_{n+1} \setminus T_n)^{1 - r/p'} + \int_{T_1} |\hat{f}|^r d\theta < \infty \end{aligned}$$

provided $ar > 1 - r/p'$, that is $a > 1/p + 1/r - 1$. In the case $p = 1$ and $r = \infty$ the theorem just states that functions in $\mathrm{Lip}(a;1) \subset L^1(G)$ have bounded Fourier transforms; this is well known.

When $p > 2$ and G is compact, we follow the above method of proof, using the property that $\|k_n * f - f\|_2 \leqslant \|k_n * f - f\|_p$. The proof is completed.

Taking $G = \boldsymbol{T}$, $V_n = \{e^{2\pi i x} \colon |x| < 2^{-n}\}$, and $T_n = \{-2^n, -2^n + 1, \ldots, 2^n - 1, 2^n\}$ in Theorem 2 we obtain the theorems of Bernstein and Szász. For $G = \boldsymbol{R}$, $V_n = (-2^{-n}, 2^{-n})$, and $T_n = [-2^n, 2^n]$ we obtain Titchmarsh's theorem. In both cases the family (k_n) is given in [3]. To extend this result to all finite-dimensional locally compact metric abelian groups we require the following lemma.

LEMMA 1. *Let G be a locally compact metric abelian group and let H be any compact subgroup of G. Suppose that* $\mathrm{Lip}(a;p)^{\wedge} \subset L^r(\Gamma_G)$ *for some choice of a, p, and r. Then the corresponding assertion holds for G/H.*

Proof. Let $f \in \mathrm{Lip}_{G/H}(a;p)$ and put $f' = f\circ\pi$, where $\pi\colon G \to G/H$ is the natural homomorphism. Then, appealing to [6], (28.54), for $a \in G$ and $p < \infty$

$$\|\tau_a f' - f'\|_p^p = \int_G |\tau_a(f\circ\pi) - f\circ\pi|^p d\lambda_G = \int_{G/H} |\tau_{\pi(a)} f - f|^p d\lambda_{G/H}$$
$$\leqslant C d_{G/H}(\pi(a), 0)^{ap} \leqslant C d_G(a, 0)^{ap};$$

thus $f' \in \mathrm{Lip}(a;p)$ and, by assumption, $\hat{f}' \in L^r(\Gamma_G)$.

Now f' is constant on cosets of H, so that $\mathrm{supp}(\hat{f}') \subset A(\Gamma_G, H)$ (the annihilator of H in Γ_G). Consequently, $\hat{f}$, being the restriction of $\hat{f}'$ to $A(\Gamma_G, H)$, belongs to $L^r(\Gamma_{G/H})$ (here we identify $\Gamma_{G/H}$ with $A(\Gamma_G, H)$).

The case $p = \infty$ is dealt with similarly.

We are now in a position to give a version of Theorem 2 for G a finite-dimensional locally compact metric abelian group. A metric d is assigned to G as in Section 1; we make use of the notation introduced there.

THEOREM 3. *Let G be a finite-dimensional locally compact metric abelian group with topological dimension m. Then, for $1 \leqslant p \leqslant 2$,* $\mathrm{Lip}(a;p)^{\wedge} \subset L^r(\Gamma_G)$, *where $a > m(1/p + 1/r - 1) \geqslant 0$.*

If, in addition, G is compact, then $\mathrm{Lip}(a;p)^{\wedge} \subset l^r(\Gamma_G)$ *for $p > 2$, where $a > m(1/r - 1/2)$.*

Proof. It is straightforward to see that, for any discrete group D, $D \times \boldsymbol{R}^{m-k} \times \Delta^\infty \times \Sigma^k$ satisfies property $P(V_n, k_n, T_n)$ with

$$V_n = \{0\} \times (-2^{-n^3}, 2^{-n^3})^{m-k} \times V_n' \times V_n'',$$

where V_n' was given in Section 1,

$$V_n'' = [\pi_B((-2^{-n^3}, 2^{-n^3}) \times A_{n-1})]^k,$$

and

$$T_n = \Gamma_D \times [-2^{n^3}, 2^{n^3}]^{m-k} \times A(\Gamma_{\Delta^\infty}, V_n') \times \{\chi_{n-2,l}\colon |l| \leqslant 2^{n^3} n!\}^k.$$

Here we identify the character group of $D \times \boldsymbol{R}^{m-k} \times \Delta^\infty \times \Sigma^k$ with $\Gamma_D \times \boldsymbol{R}^{m-k} \times \Gamma_{\Delta^\infty} \times \Gamma_{\Sigma^k}$; for details see [3] and [4]. Furthermore,

$$\delta(V_n) = 2^{-n^3}, \qquad \theta(T_n) = 2^{(n^3+1)(m-k)}[(n+1)!]^n (2^{n^3+1} n! + 1)^k,$$

and

$$\sum_{n=1}^{\infty} \delta(V_n)^\varepsilon \theta(T_{n+1} \setminus T_n)^{\varepsilon'} < \infty \qquad \text{for } \varepsilon > m\varepsilon'.$$

With minor changes to the proof of Theorem 2 we infer that Theorem 3 holds for groups of the form $D \times \boldsymbol{R}^{m-k} \times \Delta^{\infty} \times \Sigma^{k}$. Now appeal to Lemma 1 to extend the result to all groups

$$(D \times \boldsymbol{R}^{m-k} \times \Delta^{\infty} \times \Sigma^{k})/H' \cong D \times \boldsymbol{R}^{m-k} \times \big((\Delta^{\infty} \times \Sigma^{k})/H\big),$$

where $H' = \{0\} \times H$ and H is a closed 0-dimensional subgroup of $\Delta^{\infty} \times \Sigma^{k}$.

Let G be an arbitrary finite-dimensional locally compact metric abelian group. We know that $G \cong \boldsymbol{R}^{m-k} \times G_0$, where $k \leqslant m$ and G_0 contains a compact open subgroup of the form $K = (\Delta^{\infty} \times \Sigma^{k})/H$. We have already shown that Theorem 3 holds for $G' = G_0/K \times \boldsymbol{R}^{m-k} \times K$, since G_0/K is discrete, and so it remains only to show that G and G' have the same $\mathrm{Lip}(a; p)$-functions and that $L^r(\Gamma_G) = L^r(\Gamma_{G'})$.

The first assertion follows from the fact that G and G' are isometric, locally isomorphic, and have the same Haar measure. For the second notice that

$$\Gamma_G \cong \boldsymbol{R}^{m-k} \times \Gamma_{G_0} \quad \text{and} \quad \Gamma_{G'} \cong \boldsymbol{R}^{m-k} \times A(\Gamma_{G_0}, K) \times \big(\Gamma_{G_0}/A(\Gamma_{G_0}, K)\big).$$

Since K is compact and open in G_0, we infer that $A(\Gamma_{G_0}, K)$ is compact and open in Γ_{G_0}, and $\Gamma_{G_0}/A(\Gamma_{G_0}, K)$ is discrete. Hence Γ_G and $\Gamma_{G'}$ have the same Haar measure, from which the result follows.

For $p = \infty$ and $r = 1$, Theorem 3 is given in [14], Theorem 1. However, it should be noted that the metric assigned there is smaller than that given above; indeed, in defining the metric for Δ, Walker takes $\beta_n = e^{-(n+1)!}$, and for Δ^{∞}, $\beta_n = e^{-[(n+1)!]^n}$ (in our notation). It is also easy to see that, in the case $G = \Sigma_{\boldsymbol{a}}$, Theorem 3 holds when the metric on $\Sigma_{\boldsymbol{a}}$ is defined with $\beta_n = 2^{-n^2}$.

We now show that the range of values of a cannot be extended in Theorem 3 for the $\boldsymbol{a}$-adic solenoid $\Sigma_{\boldsymbol{a}}$; and, in fact, any strictly decreasing sequence (β_n) of positive numbers will serve in the definition of the metric for $\Delta_{\boldsymbol{a}}$. First, since the character group of $\Sigma_{\boldsymbol{a}}$ contains a copy of $\boldsymbol{Z}$, the duality theory for locally compact abelian groups gives the existence of a closed subgroup H of $\Sigma_{\boldsymbol{a}}$ such that $\Sigma_{\boldsymbol{a}}/H \cong \boldsymbol{T}$. Furthermore, it is straightforward to check that $H \cong \Delta_{\boldsymbol{a}}$ and that the metric given for $\Sigma_{\boldsymbol{a}}/\Delta_{\boldsymbol{a}}$ agrees with that for $\boldsymbol{T}$. Now apply Lemma 1 and the classical results of Bernstein and Szász to obtain the existence of $f \in \mathrm{Lip}(a; p)$ with $\hat{f} \notin l^r(\Gamma_{\Sigma_{\boldsymbol{a}}})$, where $1 \leqslant p \leqslant 2$ and $a = 1/p + 1/r - 1$ (respectively, $p > 2$ and $a = 1/r - 1/2$).

Finally, it is clear that Theorem 3 is sharp for G 0-dimensional.

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