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A SIMPLE PROOF OF THE CHUI-SMITH THEOREM ON LANDAU'S PROBLEM

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The purpose of this paper is to give a simple proof of the Chui-Smith theorem [1] on Landau's problem for bounded intervals.

To formulate the theorem, we will assume that f and f' are continuous on the unit interval $I = [0, 1]$ and for every $x \in I$ we have

$$\int_0^x f''(t)\,dt = f'(x) - f'(0).$$

Recall that this last requirement on f is equivalent to the absolute continuity of f' (cf. [2], Chapter 5); hence we do not assume that $f''(t)$ exists for all $t \in I$. We define

$$\|f\|_I = \operatorname*{ess\,sup}_{t\in I} |f(t)|.$$

Now we are in a position to state the theorem.

THEOREM 1 (Chui-Smith). *Let* $\|f\|_I \leqslant 1$ *and* $\|f''\|_I \leqslant A$. *Then*

$$\|f'\|_I \leqslant \begin{cases} (4+A)/2 & \text{if } 0 < A \leqslant 4, \\ 2\sqrt{A} & \text{if } A > 4. \end{cases}$$

Moreover, these inequalities are best possible.

Before proving the theorem, we should like to note that the following proof was inspired by Schoenberg [3].

Proof. Chui and Smith [1] have shown that an extremizing function f_0 is given by

$$f_0(t) = -\frac{A}{2}t^2 + \left(2+\frac{A}{2}\right)t - 1 \quad \text{if } 0 < A \leqslant 4$$

and

$$f_0(t) = \begin{cases} -\dfrac{A}{2}\left(t-\dfrac{2}{\sqrt{A}}\right)^2+1 & \text{for } 0 \leqslant t \leqslant \dfrac{2}{\sqrt{A}} \\ 1 & \text{for } t > \dfrac{2}{\sqrt{A}} \end{cases} \quad \text{if } A > 4.$$

Clearly, f_0 satisfies $\|f_0\|_I = 1$, $\|f_0''\|_I = A$, $\|f_0'\|_I = (4+A)/2$ if $0 < A \leqslant 4$, and $\|f_0'\|_I = 2\sqrt{A}$ if $A > 4$.

Now, let $\|f\|_I \leqslant 1$ and $\|f''\|_I \leqslant A$. Since f' is continuous (and, if necessary, taking $-f$ instead of f), there exists $t_0 \in I$ such that $f'(t_0) = \|f'\|_I$. Then, for any x and y with $0 \leqslant x \leqslant t_0 \leqslant y \leqslant 1$, taking the difference of the Taylor's formulas

$$f(y)-f(t_0) = (y-t_0)f'(t_0)+\int_{t_0}^{y}(y-t)f''(t)\,dt$$

and

$$f(x)-f(t_0) = (x-t_0)f'(t_0)+\int_{t_0}^{x}(x-t)f''(t)\,dt,$$

we have

$$f(y)-f(x) = (y-x)f'(t_0)+\int_{t_0}^{y}(y-t)f''(t)\,dt-\int_{x}^{t_0}(t-x)f''(t)\,dt.$$

Similarly, since

$$f_0'' = -A \quad \text{on } [0, a], \text{ where } a = \min\{1, 2/\sqrt{A}\},$$

we have

$$f_0(y-t_0)+f_0(t_0-x)-2f_0(0) = (y-x)f_0'(0)-A\left[\int_{t_0}^{y}(y-t)\,dt+\int_{x}^{t_0}(t-x)\,dt\right]$$

if $y-t_0$ and t_0-x are in $[0, a]$. On the other hand, since f_0 is concave and $f_0(a)-f_0(0) = 2$ by its definition, it follows that for some x and y with $0 \leqslant x \leqslant t_0 \leqslant y \leqslant 1$ and $y-x = a$ we have

$$f_0(y-t_0)+f_0(t_0-x)-2f_0(0) \geqslant 2.$$

By the way, note that this inequality is strict if $0 < t_0 < 1$. Hence it may be immediately proved that the extremizing functions are essentially unique; see the argument below. For bounded intervals, this result is new.

Since $f(y)-f(x) \leqslant 2$, we obtain

$$(y-x)f'(t_0)+\int_{t_0}^{y}(y-t)f''(t)\,dt-\int_{x}^{t_0}(t-x)f''(t)\,dt \leqslant 2$$
$$\leqslant (y-x)f_0'(0)-A\Big[\int_{t_0}^{y}(y-t)\,dt+\int_{x}^{t_0}(t-x)\,dt\Big]$$

or

$$(y-x)[f'(t_0)-f_0'(0)] \leqslant -A\Big[\int_{t_0}^{y}(y-t)\,dt+\int_{x}^{t_0}(t-x)\,dt\Big]-$$
$$-\Big[\int_{t_0}^{y}(y-t)f''(t)\,dt-\int_{x}^{t_0}(t-x)f''(t)\,dt\Big].$$

But since $\|f''\|_I \leqslant A$, we get

$$\Big|\int_{t_0}^{y}(y-t)f''(t)\,dt-\int_{x}^{t_0}(t-x)f''(t)\,dt\Big|$$
$$\leqslant \Big|\int_{t_0}^{y}(y-t)f''(t)\,dt\Big|+\Big|\int_{x}^{t_0}(t-x)f''(t)\,dt\Big|$$
$$\leqslant A\Big[\int_{t_0}^{y}(y-t)\,dt+\int_{x}^{t_0}(t-x)\,dt\Big];$$

thus $(y-x)[f'(t_0)-f_0'(0)] \leqslant 0$, and since $y \neq x$,

$$\|f'\|_I = f'(t_0) \leqslant f_0'(0) = \|f_0'\|_I.$$

This completes the proof.

The following theorem is an easy extension of Theorem 1 to vector-valued functions f on the interval I:

THEOREM 2. *Let $(B, |\cdot|)$ be an arbitrary Banach space and f a B-valued function defined on the interval $I = [0,1]$. Assume that $|f(t)| \leqslant 1$ for all $t \in I$, that*

$$f'(t) = \lim_{s\to t}\frac{f(s)-f(t)}{s-t}$$

exists for every $t \in I$, and that there exists a constant $A > 0$ such that

$$|f'(t)-f'(s)| \leqslant A\,|t-s| \qquad (t, s \in I).$$

Then

$$\|f'\|_I \leqslant \begin{cases} (4+A)/2 & \text{if } 0 < A \leqslant 4, \\ 2\sqrt{A} & \text{if } A > 4. \end{cases}$$

Proof. Since f' is continuous, choose $t_0 \in I$, and then, by the Hahn-Banach theorem, x^* in B^* (the dual space of B) such that

$$\|f'\|_I = |f'(t_0)| = \langle f'(t_0), x^*\rangle \quad \text{and} \quad \|x^*\| = 1.$$

Then it is easily seen that the function $h(t) = \langle f(t), x^*\rangle$ defined on the interval I is a scalar-valued function satisfying $\|h\|_I \leqslant 1$ and $\|h''\|_I \leqslant A$. Hence Theorem 1 completes the proof.

REFERENCES

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