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# ON TRACES OF EXTERIOR POWERS OF A SUM OF TWO ENDOMORPHISMS OF A PROJECTIVE MODULE

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The purpose of the present paper is to compute traces $\operatorname{tr}\Lambda^p(f+g)$ of $p$-th exterior powers of a sum of two endomorphisms $f, g$ of a finitely generated projective module over any commutative ring (with 1) of coefficients. By the use of the recursive formula given in our Main Lemma one can express $\operatorname{tr}\Lambda^p(f+g)$ as a polynomial with integral coefficients in $\operatorname{tr}\Lambda^q(f^{i_1}g^{j_1}\ldots f^{i_m}g^{j_m})$, where $q(i_1+j_1+\ldots+i_m+j_m)\leqslant p$. If the coefficient ring contains the field of rationals, then $\operatorname{tr}\Lambda^p(f+g)$ is a polynomial with rational coefficients in $\operatorname{tr}(f^{i_1}\ldots g^{j_m})$, where $i_1+\ldots+j_m\leqslant p$. This follows easily from Newton's formula for symmetric functions.

Theorems 1 and 2 will be used in a forthcoming paper in which we define characteristic series of endomorphisms of modules which admit finite projective resolutions.

**1. Endomorphisms of free modules.** Let $R$ be any commutative ring. By $R_n$ we denote the ring of all $n\times n$ matrices with coefficients in $R$. If $x\in R_n$, then we write $x=(x_{ij})$, $i,j=1,\ldots,n$, and we identify $x$ with the appropriate endomorphism of the free $R$-module on $n$ free generators $R\oplus\ldots\oplus R$.

Let $A=Z[X_{ij}^{(k)}]$, $k=0,1,\ldots,s$, $i,j=1,\ldots,n$, be the polynomial ring in $s+1$ sets of variables $X_{11}^{(k)}, X_{12}^{(k)},\ldots,X_{1n}^{(k)},\ldots,X_{nn}^{(k)}$. This ring admits natural grading: if $e_0,\ldots,e_s$ are integers, then homogenous elements of degree $e_0,\ldots,e_s$ are polynomials which are homogenous of degree $e_k$ in the $k$-th set of variables $X_{11}^{(k)},\ldots,X_{nn}^{(k)}$ for $k=0,1,\ldots,s$. Let $X^{(k)}$ be $n\times n$ matrix $X^{(k)}=(X_{ij}^{(k)})$ in the ring $A_n$. If $R$ is any commutative ring, $x^{(0)},\ldots,x^{(s)}\in R_n$, then there exists the unique homomorphism $\varphi\colon A\to R$ such that the induced homomorphism $\varphi_n\colon A_n\to R_n$ satisfies $\varphi_n(X^{(k)})=x^{(k)}$, $k=0,1,\ldots,s$. If $v\in A$ and $e_0,\ldots,e_s$ are integers, then we define

$$v(x^{(0)},\ldots,x^{(s)})=\varphi(v),$$

$$v(x^{(0)},\ldots,x^{(s)})_{e_0,\ldots,e_s}=\varphi(v_{e_0,\ldots,e_s}),$$

where $v_{e_0,\dots,e_s}$ is the homogenous component of $v$ of degree $e_0, \dots, e_s$. For instance, the trace $\operatorname{tr}\Lambda^p(X^{(0)}+\dots+X^{(s)})$ is a polynomial in $X_{ij}^{(k)}$ and

$$\varphi \operatorname{tr}\Lambda^p(X^{(0)}+\dots+X^{(s)}) = \operatorname{tr}\Lambda^p(x^{(0)}+\dots+x^{(s)})$$

and

$$\operatorname{tr}(x^{(0)}+\dots+x^{(s)})_{1,0,\dots,0} = \sum_{i=1}^{n} x_{ii}^{(0)} = \operatorname{tr} x^{(0)}.$$

We have $v(x^{(0)}, \dots, x^{(s)})_{e_0,\dots,e_s} = 0$ if some $e_k$ is negative and

$$v(x^{(0)}, \dots, x^{(s)})_{0,e_1,\dots,e_s} = v(0, x^{(1)}, \dots, x^{(s)})_{0,e_1,\dots,e_s}.$$

MAIN LEMMA. *Let $x^{(0)}, x^{(1)}, \dots, x^{(s)}$ be $n\times n$ matrices with entries in a commutative ring $R$ and let $e_0, e_1, \dots, e_s$ be non-negative integers such that $e_0+e_1+\dots+e_s = p$. Then*

$$(1)\quad \operatorname{tr}\Lambda^p(x^{(0)}+x^{(1)}+\dots+x^{(s)})_{e_0,e_1,\dots,e_s}$$
$$= \operatorname{tr}\Lambda^{e_0}(x^{(0)})\cdot\big(\operatorname{tr}\Lambda^{e_1+\dots+e_s}(x^{(1)}+\dots+x^{(s)})\big)_{e_1,\dots,e_s} -$$
$$-\sum_{m=1}^{e_0}\ \sum_{\substack{m_1,\dots,m_s\\ m_1+\dots+m_s=m}} \operatorname{tr}\Lambda^{p-m}(x^{(0)}+\dots+x^{(2s)})_{e_0-m,e_1-m_1,\dots,e_s-m_s,m_1,\dots,m_s},$$

*where $x^{(s+1)} = x^{(0)}x^{(1)}, \dots, x^{(2s)} = x^{(0)}x^{(s)}$.*

Proof. 1. We denote the number of elements of a set $t$ by $|t|$ and the set $\{1, 2, \dots, n\}$ by $\bar{n}$. If $t_0, \dots, t_s$ are disjoint subsets of $\bar{n}$, then we write

$$M(X^{(0)}, \dots, X^{(s)}; t_0, \dots, t_s) = \det(Y_{ij}),$$

where $i, j\in t_0\cup\dots\cup t_s$ and $Y_{ij} = X_{ij}^{(k)}$ if $i\in t_k$. Then for $x\in R_n$ we have

$$M(x; t) = \det(x_{j_l j_m}),\qquad l, m = 1, \dots, p,$$

if $t = \{j_1, \dots, j_p\}, j_1 < \dots < j_p$, and the well known formula for $\operatorname{tr}\Lambda^p(x$ takes the form

$$(2)\qquad \operatorname{tr}\Lambda^p(x) = \sum_{\substack{t\subset\bar{n}\\ |t|=p}} M(x; t).$$

It is easy to see that for any permutation $\pi$ of the set $0, 1, \dots, s$ we have

$$(3)\quad M(x^{(\pi(0))}, \dots, x^{(\pi(s))}; t_{\pi(0)}, \dots, t_{\pi(s)}) = M(x^{(0)}, \dots, x^{(s)}; t_0, \dots, t_s).$$

If $x^{(0)}$ is a diagonal matrix, then

$$(4)\quad M(x^{(0)}, x^{(1)}, \dots, x^{(s)}; t_0, t_1, \dots, t_s) = \Big(\prod_{i\in t_0} x_{ii}^{(0)}\Big) M(x^{(1)}, \dots, x^{(s)}; t_1, \dots, t_s).$$

Since

$$M(x^{(0)}+\ldots+x^{(s)};\bar{n}) = \det(x^{(0)}+\ldots+x^{(s)}) = \sum_{\pi\in S_n}(-1)^{\varepsilon(\pi)}\prod_{i=1}^{n}\sum_{k=0}^{s}x^{(k)}_{i,\pi(i)},$$

then for each sequence of integers $e_0,\ldots,e_s$ such that $e_0+\ldots+e_s = n$ we get

$$M(x^{(0)}+\ldots+x^{(s)};\bar{n})_{e_0,\ldots,e_s} = \sum_{\pi\in S_n}(-1)^{\varepsilon(\pi)}\sum\nolimits_{(1)}\prod_{i\in t_0}x^{(0)}_{i,\pi(i)}\cdots\prod_{i\in t_s}x^{(s)}_{i,\pi(i)}$$
$$= \sum\nolimits_{(1)} M(x^{(0)},\ldots,x^{(s)};t_0,\ldots,t_s),$$

where by $\sum_{(1)}$ we mean the sum taken over all sequences $t_0,\ldots,t_s$ of disjoint subsets of $\bar{n}$ such that $|t_k| = e_k$, $k = 0,\ldots,s$. For any subset $r\subset\bar{n}$ and any sequence of integers $e_0,\ldots,e_s$ such that $e_0+\ldots+e_s = |r|$ we get in the same way

$$(5)\qquad M(x^{(0)}+\ldots+x^{(s)};r)_{e_0,\ldots,e_s} = \sum\nolimits_{(2)} M(x^{(0)},\ldots,x^{(s)};t_0,\ldots,t_s),$$

where by $\sum_{(2)}$ we mean the sum taken over all sequences $t_0,\ldots,t_s$ of disjoint subsets of the set $r$ and such that $|t_k| = e_k$, $k = 0,\ldots,s$.

2. Let us assume that the matrix $x^{(0)}$ is diagonal. By $\sum'$ we mean the sum taken over all sequences $t_1,\ldots,t_s$ of disjoint subsets of the set $\bar{n}$ such that $|t_k| = e_k$, $k = 1,\ldots,s$, and under the sign $\sum'$ we write additional conditions imposed on sets $t_1,\ldots,t_s$. We put $t = t_1\cup\ldots\cup t_s$ for abbreviation; then by formulae (2), (5) and (4) we get

$$\operatorname{tr}\Lambda^p(x^{(0)}+\ldots+x^{(s)})_{e_0,\ldots,e_s}$$
$$= \sum_{\substack{r\subset\bar{n}\\|r|=p}} M(x^{(0)}+\ldots+x^{(s)};r)_{e_0,\ldots,e_s}$$
$$= \sum_{\substack{t_0\subset\bar{n}\\|t_0|=e_0}}\sum_{t_0\cap t=\emptyset}' M(x^{(0)},\ldots,x^{(s)};t_0,t_1,\ldots,t_s)$$
$$= \sum_{\substack{t_0\subset\bar{n}\\|t_0|=e_0}}\sum_{t_0\cap t=\emptyset}'\prod_{i\in t_0}x^{(0)}_{ii}\cdot M(x^{(1)},\ldots,x^{(s)};t_1,\ldots,t_s) = a_1-a_2,$$

where

$$a_1 = \Big(\sum_{\substack{t_0\subset\bar{n}\\|t_0|=e_0}}\prod_{i\in t_0}x^{(0)}_{ii}\Big)\Big(\sum\nolimits' M(x^{(1)},\ldots,x^{(s)};t_1,\ldots,t_s\Big),$$

$$a_2 = \sum_{\substack{t_0\subset\bar{n}\\|t_0|=e_0}}\sum_{t_0\cap t\neq\emptyset}'\prod_{i\in t_0}x^{(0)}_{ii}\cdot M(x^{(1)},\ldots,x^{(s)};t_1,\ldots,t_s).$$

Using (5) once again we get

$$a_1 = \operatorname{tr} \Lambda^{e_0}(x^{(0)}) \cdot \big(\operatorname{tr} \Lambda^{e_1+\ldots+e_s}(x^{(1)}+\ldots+x^{(s)})\big)_{e_1,\ldots,e_s}.$$

To compute $a_2$ let us remark that if subsets $t_0, t_1, \ldots, t_s$ of the set $\bar{n}$ satisfy the conditions

(i') $|t_k| = e_k$, $k = 0, 1, \ldots, s$,

(ii') $t_1, \ldots, t_s$ are disjoint,

(iii') $t_0 \cap (t_1 \cup \ldots \cup t_s) \neq \emptyset$,

then they determine subsets of the set $\bar{n}$

$$u_k = t_0 \cap t_k, \quad u_k' = t_k \setminus t_0 = t_k \setminus u_k, \quad k = 1, \ldots, s,$$
$$u' = t_0 \setminus (t_1 \cup \ldots \cup t_s),$$

which satisfy conditions

(i'') $|u_k| + |u_k'| = e_k$, $k = 1, \ldots, s$,

(ii'') $u', u_1, \ldots, u_s, u_1', \ldots, u_s'$ are disjoint,

(iii'') $1 \leqslant |u_1| + \ldots + |u_s| \leqslant e_0$.

It is easy to see that the above correspondence between sequences $t_0, \ldots, t_s$ subjected to the conditions (i')-(iii') and sequences $u', u_1, \ldots, u_s, u_1', \ldots, u_s'$ subjected to the conditions (i'')-(iii'') is one-to-one. Thus if we denote by $\sum''$ the sum taken over all sequences $u', u_1, \ldots, u_s, u_1', \ldots, u_s'$ of subsets of the set $\bar{n}$ which satisfy conditions (i'')-(iii'') and if we put $u = u_1 \cup \ldots \cup u_s$, then using formulae (4), (3) we get

$$\begin{aligned} a_2 &= \sum{}'' \prod_{i\in u'} x_{ii}^{(0)} \cdot \prod_{i\in u} x_{ii}^{(0)} \cdot M(x^{(1)}, x^{(1)}, \ldots, x^{(s)}, x^{(s)};\quad u_1, u_1', \ldots, u_s, u_s') \\ &= \sum{}'' \prod_{i\in u'} x_{ii}^{(0)} \cdot M(x^{(0)}x^{(1)}, x^{(1)}, \ldots, x^{(0)}x^{(s)}, x^{(s)}; u_1, u_1', \ldots, u_s, u_s') \\ &= \sum{}'' M(x^{(0)}, x^{(0)}x^{(1)}, x^{(1)}, \ldots, x^{(0)}x^{(s)}, x^{(s)}; u', u_1, u_1', \ldots, u_s, u_s') \\ &= \sum{}'' M(x^{(0)}, x^{(1)}, \ldots, x^{(s)}, x^{(0)}x^{(1)}, \ldots, x^{(0)}x^{(s)}; u', u_1', \ldots, u_s', u_1, \ldots, u_s). \end{aligned}$$

Let us put $m_k = |u_k|$, $k = 1, \ldots, s$, $m = m_1 + \ldots + m_s$. Then we have $|u'| = e_0 - m$, $|u_k'| = e_k - m_k$, and, using (5) once again, we get

$$a_2 = \sum_{m=1}^{e_0} \sum_{\substack{m_1,\ldots,m_s \\ m_1+\ldots+m_s=m \\ m_1\leqslant e_1,\ldots,m_s\leqslant e_s}} \operatorname{tr} \Lambda^{p-m}(x^{(0)}, \ldots, x^{(2s)})_{e_0-m, e_1-m_1, \ldots, e_s-m_s, m_1, \ldots, m_s}$$

where $x^{(s+1)} = x^{(0)}x^{(1)}, \ldots, x^{(2s)} = x^{(0)}x^{(s)}$. Thus (1) holds, because all terms in the sum in (1) for which $m_k > e_k$ vanish.

3. Let us assume that $R = C$, the field of complex numbers, and that the matrix $x^{(0)}$ is equivalent to a diagonal matrix, i.e. there exists an invertible matrix $z \in C_n$ such that the matrix $y^{(0)} = z^{-1}x^{(0)}z$ is diagonal. Let us write $y^{(k)} = z^{-1}x^{(k)}z$, $k = 1, \ldots, s$. Then

$$\operatorname{tr} \Lambda^p (x^{(0)} + \ldots + x^{(s)}) = \operatorname{tr} \Lambda^p \big(z(y^{(0)} + \ldots + y^{(s)})z^{-1}\big) = \operatorname{tr} \Lambda^p (y^{(0)} + \ldots + y^{(s)})$$

and coefficients $y_{ij}^{(k)}$ of matrices $y^{(k)}$ are linear forms in $x_{lm}^{(k)}$ and conversely. Consequently,

$$\operatorname{tr} \Lambda^p (x^{(0)} + \ldots + x^{(s)})_{e_0, \ldots, e_s} = \operatorname{tr} \Lambda^p (y^{(0)} + \ldots y^{(s)})_{e_0, \ldots, e_s}$$

and formula (1) follows by application of the lemma to matrices $y^{(0)}, \ldots y^{(s)}$.

4. Let us assume that $R = C$, the field of complex numbers. It is easy to prove that the set of all $n \times n$ matrices equivalent to diagonal matrices is dense in $C_n$. If the formula (1) holds for matrices $x^{(0)}$ in a dense subset of $C_n$ and for all $x^{(1)}, \ldots, x^{(s)} \in C_n$, then it holds for all matrices $x^{(0)}, \ldots, x^{(s)} \in C_n$.

5. Let $\mathscr{R}$ be the class of such commutative rings $R$ that Main Lemma holds for all matrices $x^{(0)}, \ldots, x^{(s)}$ in $R_n$, $n = 1, 2, \ldots$ We have proved that $C$ is in $\mathscr{R}$. It is clear that if $R$ is in $\mathscr{R}$, then any subring of $R$ and any homomorphic image if $R$ are in $\mathscr{R}$. Consequently, any finitely generated $Z$-algebra is in $\mathscr{R}$.

Let $R$ be any commutative ring and $x^{(0)}, \ldots, x^{(s)} \in R_n$. Then the subring $S$ of the ring $R$, generated by all elements $x_{ij}^{(k)}$, $k = 0, 1, \ldots, s$, $i, j = 1, \ldots, n$ and 1, is in $\mathscr{R}$. Then $R$ belongs to $\mathscr{R}$ and the proof of Main Lemma is finished.

COROLLARY 1. *There exist polynomials $w_1, w_2, \ldots$ with integer coefficients such that for any endomorphisms $f, g$ of a finitely generated free module over a commutative ring we have*

$$\operatorname{tr} \Lambda^p (f+g) = w_p(\ldots, \operatorname{tr} \Lambda^q(\mu), \ldots), \qquad p = 1, 2, \ldots,$$

*where $\mu$ varies over all monomials of the form*

$$\mu = f^{i_1} g^{j_1} \ldots f^{i_m} g^{j_m}$$

*and* $q(i_1 + j_1 + \ldots + i_m + j_m) \leqslant p$.

**2. Endomorphisms of projective modules.** All projective modules under consideration are finitely generated. The trace of an endomorphism $f\colon P \to P$ of a projective module is defined as follows (see [1]). If $F_i = P \oplus P_i$, $i = 1, 2$, are free modules, then $f_i = f \oplus 0_{P_i}$ is an endomorphism of $F_i$, $i = 1, 2$. Let $h$ be an automorphism of the free module $F_1 \oplus F_2$ defined by

$$h(p, p_1, p', p_2) = (p', p_1, p, p_2), \qquad p, p' \in P, \ p_1 \in P_1, \ p_2 \in P_2.$$

Then we have $h(f_1 \oplus 0_{F_2})h^{-1} = 0_{F_1} \oplus f_2$ and, consequently, $\mathrm{tr} f_1 = \mathrm{tr} f_2$. Thus we can put $\mathrm{tr} f = \mathrm{tr} f_1$. It is easy to see that if $\varphi: R \to S$ is a ring homomorphism and $f: P \to P$ is an endomorphism of a projective $R$-module, then $\varphi(\mathrm{tr} f) = \mathrm{tr}(f \otimes 1_S)$. It is well known (see [3]) that there exists a natural isomorphism

$$\Lambda^p(P \oplus P') \approx \bigoplus_{i+j=p} \Lambda^i(P) \otimes \Lambda^j(P') \tag{6}$$

and if $P'$ is projective, then $\mathrm{tr}(f \otimes f') = \mathrm{tr}(f)\,\mathrm{tr}(f')$ for all endomorphisms $f': P' \to P'$. It follows by (6) that $\mathrm{tr}\,\Lambda^p(f \oplus 0_{P_1}) = \mathrm{tr}\,\Lambda^p(f)$. Hence and from Corollary 1 we infer

THEOREM 1. *There exist polynomials $w_1, w_2, \ldots$ with integer coefficients such that for any endomorphisms $f, g$ of a finitely generated projective module over a commutative ring we have*

$$\mathrm{tr}\,\Lambda^p(f+g) = w_p\big(\ldots, \mathrm{tr}\,\Lambda^q(\mu), \ldots\big), \qquad p = 1, 2, \ldots,$$

*where $\mu$ varies over all monomials of the form*

$$\mu = f^{i_1} g^{j_1} \ldots f^{i_m} g^{j_m}$$

*and $q(i_1+j_1+ \ldots +i_m+j_m) \leqslant p$. We compute polynomials $w_1, w_2, \ldots$ by the use of Main Lemma.*

Let $a, b \in R_n$. Then from Main Lemma it follows that for $x = ab$ we have

$$\begin{aligned}\mathrm{tr}\,\Lambda^p(a+b)_{1,p-1} &= \mathrm{tr}\, a \cdot \mathrm{tr}\,\Lambda^{p-1}(b) - \mathrm{tr}\,\Lambda^{p-1}(a+b+x)_{0,p-2,1} \\ &= \mathrm{tr}\, a \cdot \mathrm{tr}\,\Lambda^{p-1}(b) - \mathrm{tr}\,\Lambda^{p-1}(ab+b)_{1,p-2}\end{aligned}$$

and by an obvious induction we get

$$\mathrm{tr}\,\Lambda^p(a+b)_{1,p-1} = \sum_{i=0}^{p-1} (-1)^i \mathrm{tr}(ab^i)\,\mathrm{tr}\,\Lambda^{p-1-i}(b).$$

Using this formula we get for endomorphisms $f, g$ of a projective module $P$

$$\mathrm{tr}\,\Lambda^2(f+g) = \mathrm{tr}\,\Lambda^2(f) + \mathrm{tr}(f)\,\mathrm{tr}(g) + \mathrm{tr}\,\Lambda^2(g) - \mathrm{tr}(fg),$$

$$\begin{aligned}\mathrm{tr}\,\Lambda^3(f+g) = {}& \mathrm{tr}\,\Lambda^3(f) + \mathrm{tr}\,\Lambda^2(f)\,\mathrm{tr}(g) + \mathrm{tr}(f)\,\mathrm{tr}\,\Lambda^2(g) + \mathrm{tr}\,\Lambda^3(g) - \\ & - \mathrm{tr}(fg)\,\mathrm{tr}(f+g) + \mathrm{tr}(f^2 g + fg^2)\end{aligned}$$

and using Main Lemma once again we get

$$\begin{aligned}\mathrm{tr}\,\Lambda^4(f+g) = {}& \mathrm{tr}\,\Lambda^4(f) + \mathrm{tr}\,\Lambda^3(f)\,\mathrm{tr}(g) + \mathrm{tr}\,\Lambda^2(f)\,\mathrm{tr}\,\Lambda^2(g) + \\ & + \mathrm{tr}(f)\,\mathrm{tr}\,\Lambda^3(g) + \mathrm{tr}\,\Lambda^4(g) - \mathrm{tr}(fg)\big(\mathrm{tr}\,\Lambda^2(f) + \mathrm{tr}(f)\,\mathrm{tr}(g) + \mathrm{tr}\,\Lambda^2(g)\big) - \\ & - \mathrm{tr}\,\Lambda^2(fg) + \mathrm{tr}(f)\,\mathrm{tr}(f^2 g + fg^2) + \mathrm{tr}(g)\,\mathrm{tr}(fg^2 + f^2 g) + \\ & + \mathrm{tr}(fg)\,\mathrm{tr}(fg) - \mathrm{tr}(fgfg) - \mathrm{tr}(fg^3 + f^3 g) - \mathrm{tr}(f^2 g^2).\end{aligned}$$

We denote by $f\oplus g$ the endomorphism of the module $P\oplus P$ defined by $(f\oplus g)(p,p') = (fp, gp')$, $p, p'\epsilon P$. Then by formula (6) we get

$$\begin{aligned}
\operatorname{tr}\Lambda^2(f+g) &= \operatorname{tr}\Lambda^2(f\oplus g)-\operatorname{tr}(fg),\\
\operatorname{tr}\Lambda^3(f+g) &= \operatorname{tr}\Lambda^3(f\oplus g)-\operatorname{tr}(fg)\operatorname{tr}(f\oplus g)+\operatorname{tr}\big(fg(f+g)\big),\\
\operatorname{tr}\Lambda^4(f+g) &= \operatorname{tr}\Lambda^4(f\oplus g)-\operatorname{tr}(fg)\operatorname{tr}\Lambda^2(f\oplus g)+\operatorname{tr}\big(fg(f+g)\big)\operatorname{tr}(f\oplus g)+\\
&\quad+\operatorname{tr}(fg)\operatorname{tr}(fg)-\operatorname{tr}\big(fg(f^2+fg+gf+g^2)\big)-\operatorname{tr}\Lambda^2(fg).
\end{aligned}$$

In the same way we prove that

$$\begin{aligned}
\operatorname{tr}\Lambda^5(f+g) &= \operatorname{tr}\Lambda^5(f\oplus g)-\operatorname{tr}(fg)\operatorname{tr}\Lambda^3(f\oplus g)+\operatorname{tr}\big(fg(f+g)\big)\operatorname{tr}\Lambda^2(f\oplus g)-\\
&\quad-\operatorname{tr}\big(fg(f^2+fg+gf+g^2)\big)\operatorname{tr}(f\oplus g)-\operatorname{tr}(fg)\big[\operatorname{tr}\big(fg(f+g)\big)-\\
&\quad-\operatorname{tr}(fg)\operatorname{tr}(f+g)\big]-\operatorname{tr}\Lambda^2(fg)\operatorname{tr}(f+g)+\\
&\quad+\operatorname{tr}\big(fg(f^3+f^2g+fgf+gfg+fg^2+g^3)\big).
\end{aligned}$$

THEOREM 2. *Let $f, g$ be endomorphisms of a finitely generated projective module over a commutative ring. If $fg = 0$, then*

$$\operatorname{tr}\Lambda^p(f+g) = \operatorname{tr}\Lambda^p(f\oplus g) = \sum_{i+j=p}\operatorname{tr}\Lambda^i(f)\operatorname{tr}\Lambda^j(g).$$

Proof. Let $a, b\epsilon R_n$ and $e_0+e_1 = p$. Then by Main Lemma we get for $x = ab$

$$\operatorname{tr}\Lambda^p(a+b)_{e_0,e_1} = \operatorname{tr}\Lambda^{e_0}(a)\operatorname{tr}\Lambda^{e_1}(b)-\sum_{m_1=1}^{e_0}\operatorname{tr}\Lambda^{p-m_1}(a+b+x)_{e_0-m_1,e_1-m_1,m_1}$$

and if, moreover, $ab = 0$, then

$$\operatorname{tr}\Lambda^p(a+b)_{e_0,e_1} = \operatorname{tr}\Lambda^{e_0}(a)\operatorname{tr}\Lambda^{e_1}(b)$$

and the theorem follows.

Let $x\epsilon R_n$ be a diagonal matrix and let $x_i = x_{ii}$, $i = 1,\ldots,n$. Then symmetric functions

$$\left.\begin{aligned}
s_k(x_1,\ldots,x_n) &= \sum_{i_1<\ldots<i_k} x_{i_1}\ldots x_{i_k},\\
p_k(x_1,\ldots,x_n) &= \sum_{i=1}^{n}(x_i)^k
\end{aligned}\right\}\quad k = 1,2,\ldots$$

satisfy Newton's formula (see [2])

$$\sum_{i=0}^{k-1}(-1)^i p_{k-i}s_i+(-1)^k k s_k = 0 \tag{7}$$

($s_0 = 1$, $s_k = 0$ for $k > n$). We reformulate this formula as follows:

THEOREM 3. *Let $f$ be an endomorphism of a finitely generated projective module over a commutative ring. Then*

$$\sum_{i=0}^{k-1}(-1)^i \operatorname{tr}(f^{k-i}) \operatorname{tr} \Lambda^i(f) + (-1)^k k \operatorname{tr} \Lambda^k(f) = 0, \quad k = 1, 2, \ldots \tag{8}$$

Proof. If $f$ is a diagonal endomorphism of a free module and $x_1, \ldots, x_n$ are diagonal entries, then

$$\operatorname{tr} \Lambda^k(f) = s_k(x_1, \ldots, x_n),$$
$$\operatorname{tr}(f^k) = p_k(x_1, \ldots, x_n)$$

and (8) follows by (7). We finish the proof by application of arguments used in parts 3, 4, 5 of the proof of Main Lemma.

It follows from (7) that symmetric functions $s_1, \ldots, s_n$ can be expressed as polynomials (with rational coefficients) in functions $p_1, \ldots, p_n$

$$s_k = v_k(p_1, \ldots, p_k), \quad k = 1, 2, \ldots,$$

and, similarly, by Theorem 3, it follows that

COROLLARY 2. *There exist polynomials $v_1, v_2, \ldots$ with rational coefficients such that if a commutative ring $R$ contains the field of rationals and $f$ is an endomorphism of a finitely generated projective $R$-module, then*

$$\operatorname{tr} \Lambda^p(f) = v_p\big(\operatorname{tr}(f), \ldots, \operatorname{tr}(f^p)\big), \quad p = 1, 2, \ldots$$

In particular, we have

$$\operatorname{tr} \Lambda^2(f) = \tfrac{1}{2}\big[(\operatorname{tr}(f))^2 - \operatorname{tr}(f^2)\big].$$

By Theorem 3 it follows

THEOREM 4. *There exist polynomials $u_1, u_2, \ldots$ with rational coefficients such that if a commutative ring $R$ contains the field of rationals and $f, g$ are endomorphisms of a finitely generated projective $R$-module, then*

$$\operatorname{tr} \Lambda^p(f+g) = u_p\big(\ldots, \operatorname{tr}(\mu), \ldots\big), \quad p = 1, 2, \ldots,$$

*where $\mu$ varies over all monomials of the form*

$$\mu = f^{i_1} g^{j_1} \ldots f^{i_m} g^{j_m}$$

*and* $i_1 + j_1 + \ldots + i_m + j_m \leqslant p$.

If a ring $R$ contains the field of rationals, then Theorem 2 for endomorphisms of projective $R$-modules follows from Theorem 3 by easy induction on $p$.

## *REFERENCES*

[1] N. Bourbaki, *Éléments de Mathématique. Algèbre*, Chap. 2, Paris 1962.
[2] — *Éléments de Mathématique. Algèbre*, Chap. 5, Paris 1950.
[3] W. H. Greub, *Multilinear algebra*, Berlin 1967.

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