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BOREL-DENSE BLACKWELL SPACES ARE STRONGLY BLACKWELL

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0. Introduction. In their monograph on Borel structures, Bhaskara Rao and Rao (1981) posed the problem (P4) of whether every Blackwell space is strongly Blackwell. We answer this question in the affirmative for a particular class of Blackwell spaces, namely those Borel-dense (i.e. with totally imperfect complement) in some standard space; in particular, we prove that the constructions given in Orkin (1972) and Section 9 of Bhaskara Rao and Rao (1981) both produce exactly these spaces. Additionally, they are characterised as "Borel-dense of order 2", as defined below.

For other results on Blackwell spaces, the reader is referred to the works of Maitra (1970), Sarbadhikari (1973) and Ramachandran (1975); the latter gives certain relations with foundational probability.

1. Preliminaries. We work exclusively with *separable spaces*, i.e. measurable spaces $(X, \mathscr{B})$ whose σ-algebra $\mathscr{B}$ is countably generated (c.g.) and separates points of X. Often, the notation of a σ-algebra is suppressed: the space is called X only, and when needed, its measurable structure is indicated by $\mathscr{B} = \mathscr{B}(X)$. If $\mathscr{C}$ is a sub-σ-algebra of $\mathscr{B}(X)$, and $A \subset X$, then we use the notations:

$$\mathscr{B}(A) = \{B \cap A \colon B \in \mathscr{B}(X)\} \quad \text{and} \quad \mathscr{C}(A) = \{C \cap A \colon C \in \mathscr{C}\}.$$

A separable space $(S, \mathscr{B})$ is *standard* if there is a complete separable metric topology on S for which $\mathscr{B}$ is the corresponding Borel structure. If $\mathscr{C}$ and $\mathscr{D}$ are c.g. sub-σ-algebras of $\mathscr{B}(S)$, then say that *$\mathscr{C}$ is proper in $\mathscr{D}$* when:

1) $\mathscr{C} \subset \mathscr{D}$, and

2) there are uncountably many atoms of $\mathscr{C}$ that are not atoms of (i.e. are "split" by) $\mathscr{D}$.

A separable space X is a *Blackwell space* if whenever $\mathscr{C}$ is a c.g. sub-σ-algebra of $\mathscr{B}(X)$ that separates points, then $\mathscr{C} = \mathscr{B}(X)$. A separable X is *strongly Blackwell* if whenever $\mathscr{C} \subset \mathscr{D}$ are c.g. sub-σ-algebras of $\mathscr{B}(X)$ with the same atoms, then $\mathscr{C} = \mathscr{D}$. If X is a subset of a standard space S, then X is *(*)-Blackwell in S* if whenever $\mathscr{C}$ is a c.g. sub-σ-algebra of $\mathscr{B}(S)$ that is proper in $\mathscr{B}(S)$, then there is some atom C of $\mathscr{C}$ such that $C \cap X$ contains at least two distinct points (i.e. $\mathscr{C}$ does not separate points of X). A subset X is *strongly (*)-Blackwell in S* if whenever $\mathscr{C}$ and $\mathscr{D}$ are c.g. sub-σ-algebras of $\mathscr{B}(S)$ with $\mathscr{C}$ proper in $\mathscr{D}$, then there is some atom C of $\mathscr{C}$ and two distinct points in $C \cap X$ that are separated by $\mathscr{D}$.

It is not hard to see that the following lattice of implications obtains:

$$\begin{tikzcd}
(X \text{ strongly } (*)\text{-Blackwell in } S) \arrow[r, Rightarrow] \arrow[d, Rightarrow] & (X \text{ strongly Blackwell}) \arrow[d, Rightarrow] \\
(X\ (*)\text{-Blackwell in } S) \arrow[r, Rightarrow] & (X \text{ Blackwell})
\end{tikzcd}$$

Cf. Bhaskara Rao and Rao (1981) Chapter 2, Sections 8 and 9, esp. Proposition 9.

If S is any set and $s \in S$, then by a *1-slice of $S \times S$ over the point s* we mean a set of the form $\{s\} \times S$ or $S \times \{s\}$; if s is not specified, then we refer simply to a *1-slice of $S \times S$*. If $B \subset S \times S$, then by a *1-section of B* we mean the intersection of B with a 1-slice of $S \times S$; if C is a 1-slice of $S \times S$ over the point s, then $B \cap C$ is a *1-section of B over the point s*. A 1-section is naturally identified with its one-one projection on one of the S factors. A subset B of $S \times S$ is *symmetric* if $(s, t) \in B$ implies $(t, s) \in B$.

Let S be a standard space; a subset X of S is *Borel-dense of order* 1 *in S* (or simply *Borel-dense in S*) if $S \setminus X$ contains no uncountable members of $\mathscr{B}(S)$, or what is equivalent, no uncountable analytic sets. Also, X is *Borel-dense of order* 2 *in S* if whenever $B \in \mathscr{B}(S \times S)$ is a subset of $(S \times S) \setminus (X \times X)$, then B is contained in a countable union of 1-slices of $S \times S$ over points in $S \setminus X$. It is not hard to see that Borel-density of order 2 implies that of order 1. A more complete study of Borel densities of order n is (Shortt (1984)).

Example 1. A conventional argument using transfinite induction establishes the existence of a Borel-dense subset X of the real numbers $\boldsymbol{R}$ such that $X \times X$ does not meet the line $y = -x$ in the plane $\boldsymbol{R}^2$. Thus X is Borel-dense of order 1, but not of order 2.

LEMMA 1. *Let S be a standard space; if $X \subset S$ is (*)-Blackwell in S, then X is Borel-dense (of order 1) in S.*

Proof. If $B \subset S \setminus X$ is an uncountable member of $\mathscr{B}(S)$, then there is an isomorphism j of B onto $B \times B$; let f_0: $B \to B$ be the map j followed by projection onto the first factor of $B \times B$. Define f: $S \to S$ by

$$f(s) = \begin{cases} f_0(s) & \text{for } s \in B, \\ s & \text{for } s \in S \setminus B \end{cases}$$

and put $\mathscr{C} = \mathscr{B}_f = \{f^{-1}(A)\colon A \in \mathscr{B}(S)\}$. Then $\mathscr{C}$ is a c.g. σ-algebra proper in $\mathscr{B}(S)$ and separating points of X. The lemma follows by contraposition. Q.E.D.

It is the purpose of our main theorem below to bring attention to the fact that the notions of strongly (*)-Blackwell, (*)-Blackwell, and second-order Borel-density coincide for subsets X of a standard space S; moreover, such subsets X are precisely those Blackwell spaces Borel-dense in S. Thus the notions of Blackwell space and strongly Blackwell space coincide for Borel-dense sets. Before proceeding, we require the use of four more lemmas.

LEMMA 2. *Let E and F be analytic spaces and let A be an analytic subset of $E \times F$. If $A(y) = \{x \in E\colon (x, y) \in A\}$ denotes the 1-section of A over the point y, then $\{y \in F\colon A(y)$ is uncountable$\}$ is an analytic subset of F.*

Proof. This theorem is originally due to Mazurkiewicz and Sierpiński (1924) and has been generalised by Hoffmann-Jørgensen (1970), III.6.1.

LEMMA 3. *Let A be a standard subset of the product $E \times F$ of analytic spaces E and F. If the 1-sections $A(x) = \{y \in F\colon (x, y) \in A\}$ are countable for all x in E, then there exist standard sets $B_n \subset E$ ($n = 1, 2, \ldots$), and measurable mappings $f_n\colon B_n \to F$ such that:*

1) *$f_n(x) \neq f_m(x)$ for all x in $B_n \cap B_m$ and $n \neq m$,*

and

2) *$A = \bigcup_{n=1}^{\infty} G(f_n)$, where $G(f_n)$ is the graph of f_n.*

Proof. This theorem is essentially due to Lusin (1930) p. 243; a proof is to be found in Hoffmann-Jørgensen (1970), III.6.7.

Let E and F be separable spaces and let S be an uncountable standard subset of $E \times F$. Given x_0 in E and y_0 in F, define the 1-sections

$$S_1(x_0) = \{y \in F\colon (x_0, y) \in S\},$$
$$S_2(y_0) = \{x \in E\colon (x, y_0) \in S\}.$$

LEMMA 4. *Suppose that for each $x \in E$ and $y \in F$, one has $S_1(x)$ and $S_2(y)$ countable; then there is an uncountable standard subset S_0 of E and a one-one measurable function $f\colon S_0 \to F$ whose graph $G(f)$ is contained in S.*

Proof. Using Lemma 3, we find (for $n = 1, 2, \ldots$) standard subsets $B_n \subset E$ and measurable mappings $f_n\colon B_n \to F$ so that $S = \bigcup_{n=1}^{\infty} G(f_n)$; select n so that $G(f_n)$ is uncountable. Notice that since $g_n\colon B_n \to S$ defined by $g_n(x) = (x, f_n(x))$ is one-one and measurable, its range $G(f_n)$ belongs to $\mathscr{B}(S)$ and so is standard.

Apply Lemma 3 once more, this time to the set $G(f_n)$, using the fact that its "horizontal" sections are countable. There are, for $m = 1, 2, \ldots$, standard subsets $C_m \subset F$ and measurable mappings $g_m: C_m \to E$ so that $G(f_n) = \bigcup_{m=1}^{\infty} G(g_m)$; select m so that $G(g_m)$ is uncountable.

Since each "vertical" section of $G(f_n)$, hence of $G(g_m)$ is a singleton, $g_m: C_m \to B_n$ is one-one and so bimeasurable. We may put $S_0 = g_m(C_m)$ and $f = g_m^{-1}$ on S_0: Q.E.D.

The same argument shows that S is the countable union of such graphs.

LEMMA 5. *Let X be a subset of a standard space S such that X is Borel-dense of order* 1 *but not of order* 2 *in S; then there is a measurable automorphism g of S onto itself such that*

a) *$g \circ g$ is the identity map on S, and*

b) *the set $T = \{(s, g(s)): g(s) \neq s\}$ is uncountable and does not meet $X \times X$.*

Proof. If X is not Borel-dense of order 2 in S, then there is some $B \in \mathscr{B}(S \times S)$ with $B \subset (S \times S) \setminus (X \times X)$ but such that B is not contained in a countable union of 1-slices of $S \times S$. B may be chosen symmetric and, assuming X is order-one dense in S, such that B does not meet the diagonal Δ of $S \times S$: in any case *via* projection onto one co-ordinate, $B \cap \Delta$ would be isomorphic with a standard subset of $S \setminus X$ and so would be at most countably infinite.

Consider now the 1-sections of B: over points of X, these are standard subsets of $S \setminus X$ and so, by the Borel-density of X, are countable. The set of all points in S for which these 1-sections are uncountable is, by Lemma 2, an analytic subset of $S \setminus X$ and so is countable. Subtract from B the 1-slices of $S \times S$, in each co-ordinate, over the points in this countable set. What remains of B is a subset B_0 of $(S \times S) \setminus (X \times X)$ such that:

(i) B_0 is symmetric;

(ii) $B_0 \in \mathscr{B}(S \times S)$;

(iii) $B_0 \cap \Delta = \emptyset$;

(iv) each 1-section of B_0 is countable;

(v) B_0 is uncountable.

Using the isomorphism theorem for standard (or "absolute") Borel spaces, we consider S as a Borel subset of the real line with its usual order and metric structure. Define

$$B_- = \{(s, t) \in B_0: s > t\}, \quad B_+ = \{(s, t) \in B_0: s < t\},$$

disjoint, uncountable standard sets with $B_0 = B_- \cup B_+$.

By Lemma 4, there are uncountable standard subsets D and R of S and an isomorphism h of D onto R whose graph H is a subset of B_-: then $h(s) < s$ for all s in D, and there is some $\varepsilon > 0$ such that

$$D(\varepsilon) = \{s \in D: h(s) < s - \varepsilon\}$$

is uncountable. Then there is some open interval N of length ε such that $D_0 = N \cap D(\varepsilon)$ is uncountable. Whenever s and t are elements of D_0, then $h(s) < t$: so $D_0 \cap h(D_0) = \emptyset$.

Define $g\colon S \to S$ by the rule

$$g(s) = \begin{cases} h(s) & \text{if } s \in D_0, \\ h^{-1}(s) & \text{if } s \in h(D_0), \\ s & \text{otherwise.} \end{cases}$$

Then g is an automorphism of S such that $g \circ g$ is the identity map. Also, $T = \{(s, g(s))\colon g(s) \neq s\}$ is an uncountable subset of B_0 and so does not meet $X \times X$. Q.E.D.

The construction bears comparison with Corollary 2 of Shortt (1984).

2. The Principal Result.

THEOREM. *Let X be a subset of a standard space S; then the following statements are equivalent:*

1) *X is Borel-dense of order 2 in S,*
2) *X is strongly (*)-Blackwell in S,*
3) *X is (*)-Blackwell in S,*
4) *X is a Blackwell space and is Borel-dense in S.*

Proof. 1) *implies* 2). Assume that $\mathscr{C}$ and $\mathscr{D}$ are c.g. sub-σ-algebras of $\mathscr{B}(S)$ with $\mathscr{C}$ proper in $\mathscr{D}$. Let f and g be Marczewski functions for $\mathscr{C}$ and $\mathscr{D}$, respectively, and consider the set

$$T = \{(s, t) \in S \times S\colon g(s) \neq g(t) \text{ and } f(s) = f(t)\}.$$

T is a member of $\mathscr{B}(S \times S)$ which, since $\mathscr{C}$ is proper in $\mathscr{D}$, is not contained in a countable union of 1-slices of $S \times S$. If X is second-order Borel-dense in S, then $X \times X$ must intersect T; thus X is strongly (*)-Blackwell in S.

2) *implies* 3). Trivial.

3) *implies* 1). Assume X is (*)-Blackwell in S; then by Lemma 1, X is Borel-dense (of order 1) in S. If, however, X is not second-order Borel-dense in S, then by Lemma 5, there is a measurable automorphism $g\colon S \to S$ such that:

a) $g \circ g$ is the identity map on S;

b) the set $T = \{(s, g(s))\colon g(s) \neq s\}$ is uncountable and does not meet $X \times X$.

Since S is isomorphic with some Borel subset of the real line, it makes sense to speak of a linear ordering $\leqslant$ on S that respects (and generates) the Borel structure $\mathscr{B}(S)$. Having fixed such an ordering, we now define $f\colon S \to S$

by $f(s) = s \wedge g(s)$, the minimum of s and $g(s)$. So

$$\begin{aligned} T &= \{(s, t)\colon s \neq t,\ g(s) = t\} \\ &= \{(s, t)\colon s \neq t,\ s \wedge g(s) = t \wedge g(t)\} \\ &= \{(s, t)\colon s \neq t, f(s) = f(t)\} \end{aligned}$$

is an uncountable member of $\mathscr{B}(S \times S)$ not meeting $X \times X$.

Consider $\mathscr{B}_f = \{f^{-1}(B)\colon B \in \mathscr{B}(S)\}$; the atoms of $\mathscr{B}_f$ are given by

$$f^{-1}(t) = \begin{cases} \{t\} & \text{if } t = g(t), \\ \emptyset & \text{if } t > g(t), \\ \{t, g(t)\} & \text{if } t < g(t), \end{cases}$$

so that $\mathscr{B}_f$ is c.g. and proper in $\mathscr{B}(S)$, and, since $X \times X$ does not meet T, $\mathscr{B}_f(X)$ is separable. Therefore X is not (*)-Blackwell in S.

3) *implies* 4). Immediate from Lemma 1.

4) *implies* 3). Suppose that $\mathscr{C}$ is a c.g. sub-σ-algebra of $\mathscr{B}(S)$ such that $\mathscr{C}(X)$ is separable. Let f be a Marczewski function for $\mathscr{C}$; if X is a Blackwell space, then $\mathscr{C}(X) = \mathscr{B}(X)$, and f is an isomorphism when restricted to X. Thus f is an isomorphism on some member S_0 of $\mathscr{B}(S)$ containing X. If X is Borel-dense in S, then $S \setminus S_0$ is countable. This means that no c.g. sub-σ-algebra $\mathscr{C}$ of $\mathscr{B}(S)$ can be proper in $\mathscr{B}(S)$ and still separate points of X, i.e. X is (*)-Blackwell in S. Q.E.D.

COROLLARY. *A Borel-dense subset of a standard space is Blackwell if and only if it is strongly Blackwell.*

Remark. The following two statements are implied by Corollary 5 and Proposition 13, respectively, in Shortt (1984):

1) If X is Borel-dense and universally measurable in S, then X is a strongly Blackwell space.

2) If X is a non-Borel (*)-Blackwell subset of S, then X is not isomorphic with the product of any two uncountable spaces.

Remark. Returning to example 1, we see that the function $g\colon \boldsymbol{R} \to \boldsymbol{R}$ guaranteed to exist in Lemma 5 may be taken to be $g(s) = -s$. Referring to the proof that $3) \Rightarrow 1)$ in the theorem, we find $f(s) = s \wedge g(s) = s \wedge (-s) = -|s|$; then $\mathscr{B}_f$ consists of those Borel sets of $\boldsymbol{R}$ symmetric about zero, a c.g. σ-algebra proper in $\mathscr{B}(\boldsymbol{R})$. Here $\mathscr{B}_f(X)$ is separable, but cannot coincide with $\mathscr{B}(X)$: were it so, f would be an isomorphism on X and therefore on all but countably many points of $\boldsymbol{R}$ (by Borel-density), clearly a contradiction.

Remark. The (*)-Blackwell sets discussed above are easily seen to be precisely those satisfying condition 2* in the paper of Orkin (1972). Our theorem above shows that this actually coincides with the seemingly stronger construction due to Ryll-Nardzewski and presented in Sarbadhikari (1973).

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