COLLOQUIUM MATHEMATICUM

VOL. XLI 1979 FASC. 2

A LOWER BOUND FOR THE i-TH TOTAL ABSOLUTE CURVATURE OF AN IMMERSION

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We consider the i-th total absolute curvature $TA_i(f, \beta)$ of order β of an immersion $f: M \to E^{n+m}$ of a closed n-dimensional manifold into a Euclidean space. In [1] Chen has proved the inequality $TA_i(f, n/i) \geqslant 2$ generalizing classical inequalities of Fenchel-Borsuk-Willmore-Chern-Lashof. In this note we give a shorter proof of this theorem. Moreover, it turns out that in the case $i\beta \neq n$, for a fixed M, the infimum of $TA_i(f, \beta)$ over the class of all immersions f into Euclidean spaces is zero. In a special case this has been shown by Friedrich [3].

Let $f: M^n \to E^{n+m}$ be a C^∞-immersion of a closed manifold of dimension n into a Euclidean space. Let N^1 denote the total space of the unit normal bundle of f with volume form σ. Each $e \in N_p^1$ induces the *second fundamental tensor* in the direction e as an endomorphism $A(e): T_pM \to T_pM$ of the tangent space. Let $k_1(e), \ldots, k_n(e)$ be the eigenvalues of $A(e)$. Then

$$K_i(e) := \binom{n}{i}^{-1} \sum_{j_1<\ldots<j_i} k_{j_1}(e)\ldots k_{j_i}(e)$$

is called the *i-th curvature* in the direction e.

The integral

$$TA_i(f, \beta) := \frac{1}{c_{m+i\beta-1}} \int_{N^1} |K_i(e)|^\beta \sigma,$$

defined by Chen, is called the *i-th total absolute curvature of order β* of f, where c_k denotes the volume of the unit sphere S^k. Note that $TA_i(f, \beta)$ does not depend on the codimension, which means that $TA_i(f, \beta) = TA_i(j\circ f, \beta)$ for each linear embedding $j: E^{n+m} \to E^{n+m+k}$ ([5], Lemma 1).

Some facts.

(1) $TA_n(f, 1) \geqslant \beta(M) \geqslant 2$, where $\beta(M)$ denotes the sum of the Betti numbers of M with coefficients in a field. The equation $TA_n(f, 1) = 2$

stands exactly for convex hypersurfaces in an $(n+1)$-dimensional linear subspace (cf. [2]).

(2) Let $f\colon M^2 \to E^{2+m}$ and $\bar{f}\colon \bar{M}^2 \to E^{2+m}$ be immersions of closed surfaces, let $f\times\bar{f}\colon M\times\bar{M} \to E^{4+2m}$ be the product immersion, and let β_1 and $\bar{\beta}_1$ denote the first Betti numbers of M and $\bar{M}$, respectively, with coefficients in a field (in the non-orientable case in Z_2). Then

$$T_2(f\times\bar{f}, 2) \geqslant 2+\frac{2}{9}\,\frac{m+2}{m}+\frac{2}{9m}(\beta_1+\bar{\beta}_1)+\frac{m+1}{9m}\beta_1\bar{\beta}_1$$

with equality exactly if $m = 1$ and $f(M)$ and $\bar{f}(\bar{M})$ are round spheres with the same radii (cf. [5], Theorem 3).

THEOREM. *Let $f\colon M^n \to E^{n+m}$ be an immersion of the closed connected manifold M.*

(i) *If $i\beta = n$, then $TA_i(f, \beta) \geqslant 2$, and in the case $1 \leqslant i < n$ the equality stands exactly for the round hypersphere in an $(n+1)$-dimensional linear subspace* (see [1]).

(ii) *If $i\beta \neq n$, then*

$$\inf_f TA_i(f, \beta) = 0.$$

Proof. Each real number $\lambda \in (0, \infty)$ induces an immersion $f_\lambda\colon M \to E^{n+m}$ defined by $f_\lambda(p) := \lambda f(p)$. Obviously, the second fundamental tensors A_λ and A of f_λ and f, respectively, satisfy the equation $A = \lambda A_\lambda$. Hence we get directly the equation

$$TA_i(f_\lambda, \beta) = \lambda^{n-i\beta}\, TA_i(f, \beta).$$

Consequently,

$$\lim_{\lambda\to\infty} TA_i(f_\lambda, \beta) = 0 \quad \text{for } n < i\beta$$

and

$$\lim_{\lambda\to 0} TA_i(f_\lambda, \beta) = 0 \quad \text{for } n > i\beta,$$

which proves part (ii) of the assertion. (Using the theory of critical points the proof of (ii) in the case $i = n$ is much more complicated (cf. [3]).)

To prove part (i) let $U \subseteq N^1$ denote the set of all normal vectors $e \in N^1$ such that all eigenvalues of $A(e)$ have the same sign. Using the inequality $|K_i(e)|^{1/i} \geqslant |K_{i+1}(e)|^{1/(i+1)}$ for all i (cf. [4], p. 104), we get

$$\text{(1)}\quad \int_{N^1} |K_i(e)|^\beta\sigma \geqslant \int_U |K_i(e)|^\beta\sigma = \int_U |K_i(e)|^{n/i}\sigma \geqslant \int_U |K_n(e)|\,\sigma \geqslant 2c_{n+m-1}.$$

The last inequality holds by the well-known argument that the integral over $|K_n|$ is the volume of the image of the Gauss map, and in our case the whole sphere is at least twice covered by U.

For $f(M)$ being a round hypersphere, in an $(n+1)$-dimensional linear subspace we have $TA_i(f, \beta) = 2$ trivially.

Now assume that $1 \leqslant i < n$ and $TA_i(f, \beta) = 2$. Then inequality (1) becomes an equality at every step, especially

$$|K_i(e)|^\beta = \begin{cases} 0 & \text{for } e \in N^1 \setminus U, \\ |K_n(e)| & \text{for } e \in U, \end{cases}$$

whence $k_1(e) = \ldots = k_n(e)$ (cf. [4], p. 104) or

$$A(e) = g(e) \cdot \mathrm{id} \tag{2}$$

with $g(e)$ being a scalar function.

Let $U' \subseteq N$ be the set of all normals in the total space of the normal bundle satisfying (2) with $g(e) \neq 0$. Then U' is not empty because of the inequality $TA_i(f, \beta) \geqslant 2$. For $e \in U'$ with base point p, $U' \cap N_p$ is an open set in the normal space N_p. Hence (2) holds for a linear basis of N_p. Consequently, by the linearity in e of expression (2), equality (2) holds for the whole space N_p (where $g(e)$ may be zero, of course).

Now, let π: $N^1 \to M$ be the canonical projection. Then $\pi(U')$ is open in M and $f|_{\pi(U')}$ is totally umbilical. Hence Theorem 26 in [6] states that the image of each component of $\pi(U')$ under f lies in some round hypersphere in an $(n+1)$-dimensional linear subspace of E^{n+m}. By construction, each such component is open in M. Now, let q be one of its limit points and assume that $q \notin \pi(U')$. Then, for each $e \in N_q$, $|K_i(e)|^\beta = 0$, contradicting the fact that q is the limit point of spherical points. Hence $\pi(U')$ is topologically closed and, therefore, $\pi(U') = M$. This means that $f(M)$ is a round hypersphere in an $(n+1)$-dimensional linear subspace.

The proof of (i) given in [1], Theorems 1 and 2, is much more complicated because of the use of Lemmas 2 and 4, which seems to be unnecessary.

In view of Facts (1) and (2), it seems to be an interesting problem to find lower bounds for $TA_i(f, n/i)$ depending on the topology of M (**P 1073**)

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Reçu par la Rédaction le 13. 10. 1976 ;
en version modifiée le 9. 11. 1977