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ADJOINING CONJUGATING ELEMENTS TO FINITE GROUPS

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If φ is an isomorphism of subgroups A and B of a group G, then the group K with presentation $\langle G, t\colon t^{-1}at = \varphi(a)$ for all $a \in A\rangle$ (the HNN extension of G relative to φ) is known to have many nice properties. In particular, a normal form for elements of K is given by Britton's lemma (see [2], Chapter IV). One measure of this niceness is the way the conjugating element t interacts with subgroups of G. For example:

(*) If $H \subset G$ intersects A and B trivially, then $\langle H, t\rangle \cap G = H$.

If G is finite, it is easy to obtain a finite group $\langle G, t\rangle$ which satisfies the relations of K, and one might ask whether any of the nice properties of K can be carried over to such a finite group. Here* we will prove such a theorem, our concern being to satisfy (*) in a finite group.

THEOREM 1. *Suppose G is a finite group, $A, B \subset G$, and*

(i) *$\varphi\colon A \to B$ is an isomorphism which extends to $\tau \in \mathrm{Aut}\langle A, B\rangle$ with $|\tau| = 2$.*

There exists a finite group $\langle G, t\rangle$ with $|t| = 2$ such that t induces φ on A by conjugation and, for all $H \subset G$ such that

(ii) *$H \cap \langle A, B\rangle$ is τ-invariant,*

we have $\langle H, t\rangle \cap G = H$.

Notice that condition (ii) is also necessary for the conclusion since $\langle A, B\rangle$ is t-invariant.

For our main theorem, we will further restrict the subgroups A and B.

THEOREM 2. *Suppose G is a finite group, $A, B \subset G$ and*

(i) *$|A| = |B| = 2$.*

There exists a finite group $\langle G, t\rangle$ with $|t| = 4$ such that $A^t = B$ and, for all $H \subset G$ such that $A \cap H = B \cap H = 1$ and

(ii) *either $\langle A, H\rangle \cap B = 1$ or $\langle B, H\rangle \cap A = 1$,*

we have $\langle H, t\rangle \cap G = H$.

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This is probably an initial case of a result in which assumptions (i) and (ii) are weakened or eliminated. These restrictions seem to arise from our method of proof and not from any essential feature of the situation.

Bounds for the order of $\langle G, t\rangle$ can be extracted from the proof, but would be very large because permutational products are used.

The proofs can be motivated by viewing the HNN extension K as a split extension by $\langle t\rangle$ of an iterated free amalgamated product P obtained from copies G_n ($n \in \mathbf{Z}$) of G with amalgamations $B_n = A_{n+1}$ (via φ, i.e., $a_{n+1} = \varphi(a)_n$ for all $a \in A$) on which t acts so that $t^{-1}g_n t = g_{n+1}$ for all $g \in G$ and $n \in \mathbf{Z}$ (where $g_i \in G_i$ corresponds to $g \in G$ under a canonical map). To obtain a group $\langle G, t\rangle$ in which t has finite order n, we must replace the infinitely iterated amalgam P by a finite cyclic chain of amalgams in which $B_n = A_1$ (via φ). This reduces the problem of adjoining to G a "nice" conjugating element t so that $\langle G, t\rangle$ is finite to a problem of performing "nice" amalgamations of finite groups, which can be done using permutational products. Assumption (i) of both theorems is needed in our proofs to "close" the finite chain of amalgamations of $G_1, \ldots, G_n$ at $n = 2$ or $n = 4$.

Before proceeding with the details we will give our amalgamating tool.

An *amalgam*

$$\mathfrak{A} = \begin{array}{ccc} F & & H \\ & \diagdown\!\diagup & \\ & E & \end{array}$$

is the union of two groups F and H, $\mathfrak{A} = F \cup H$, which meet in a common subgroup $E = F \cap H$. An amalgam

$$\mathfrak{A}_0 = \begin{array}{ccc} F_0 & & H_0 \\ & \diagdown\!\diagup & \\ & E_0 & \end{array}$$

is a *subamalgam* of $\mathfrak{A}$ if $F_0 \leqslant F$, $H_0 \leqslant H$ and $F_0 \cap E = H_0 \cap E = E_0$.

Definition. If $\mathfrak{A}$ is an amalgam and G is a group generated by $\mathfrak{A}$, then G has the *subamalgam property* (s.p.) if $\langle \mathfrak{A}_0\rangle \cap \mathfrak{A} = \mathfrak{A}_0$ for all subamalgams $\mathfrak{A}_0$ of $\mathfrak{A}$. In the above notation, this means that, in G, $\langle F_0, H_0\rangle \cap F = F_0$ and $\langle F_0, H_0\rangle \cap H = H_0$.

SUBAMALGAM LEMMA ([1], p. 226). *Suppose $\mathfrak{A}$ is an amalgam of finite groups. There exists a finite group $G = \langle \mathfrak{A}\rangle$ satisfying the s.p.*

The proof of this lemma uses permutational products, but no further use of permutational products will be made in this paper.

Suppose $\mathfrak{A}$ is an amalgam. $\mathrm{gp}_*(\mathfrak{A})$ is the free amalgamated product of $\mathfrak{A}$. If A_1 and A_2 are groups generated by $\mathfrak{A}$, we write $f\colon A_1 \xrightarrow{\mathfrak{A}} A_2$ if f is a homomorphism from A_1 to A_2 and $f(a) = a$ for all $a \in \mathfrak{A}$.

We will need

LEMMA 1. *Suppose $\mathfrak{A}$ is an amalgam.*

(i) *If A is any group generated by $\mathfrak{A}$, then there exists*

$$f\colon \mathrm{gp}_*(\mathfrak{A}) \xrightarrow{\mathfrak{A}} A.$$

(ii) *If A_1 and A_2 are groups generated by $\mathfrak{A}$, $f\colon A_2 \xrightarrow{\mathfrak{A}} A_1$ and A_1 has the s.p., then A_2 also has the s.p.*

The first part is simply the universal mapping property of $\mathrm{gp}_*(\mathfrak{A})$, while (ii) is checked routinely — any violation of the s.p. in A_2 is preserved under f.

Proof of Theorem 1. Let G_1 be a copy of G. By hypothesis (i) of the theorem, the subgroups $\langle A, B\rangle \subset G$ and $\langle A_1, B_1\rangle \subset G_1$ are isomorphic by an extension of the map $a \to \varphi(a)_1$ ($a \in A$) and $b \to \varphi^{-1}(b)_1$ ($b \in B$), i.e., the map which transposes A and B via φ. So, we will arrange that G and G_1 intersect in this manner, i.e.,

$$\mathfrak{A} = \begin{array}{ccc} G & & G_1 \\ & \diagdown\ \diagup & \\ \langle A, B\rangle & = & \langle A_1, B_1\rangle \end{array}$$

is an amalgam where

(1) $\quad a = \varphi(a)_1$ for all $a \in A \quad$ and $\quad b = \varphi^{-1}(b)_1$ for all $b \in B$.

Now $\mathrm{gp}_*(\mathfrak{A})$ has an automorphism τ of order 2 such that $\tau(g) = g_1$ for all $g \in G$ because this map and its inverse preserve relations (1). Let $P = \langle\mathfrak{A}\rangle$ be a finite group satisfying the s.p. and let $f\colon \mathrm{gp}_*(\mathfrak{A}) \to P$ with $N = \ker(f)$. Put

$$P_1 = \mathrm{gp}_*(\mathfrak{A})/N \cap N^\tau = \langle\mathfrak{A}\rangle.$$

Thus, P_1 is finite, there exists $f_1\colon P_1 \xrightarrow{\mathfrak{A}} P$, and hence P_1 has the s.p. by Lemma 1, and P_1 has an automorphism, which we also call τ, such that $|\tau| = 2$ and $\tau(g) = g_1$ for all $g \in G$.

We claim that $\langle G, t\rangle = P_1\langle t\rangle$, where $|t| = 2$ and t induces τ on P_1, is the desired group. Note that $G_1 = G^t$, so $P_1 = \langle G, G_1\rangle \subset \langle G, t\rangle$. From relations (1) we have $\tau(a) = t^{-1}at = \varphi(a)$ for all $a \in A$ so that t does induce φ on A as required, and $\langle A, B\rangle = \langle A_1, B_1\rangle$ is t-invariant.

Suppose $H \subset G$ and $H \cap \langle A, B\rangle$ is invariant under τ. We must check that $\langle H, t\rangle \cap G = H$, that is, since $\langle H, t\rangle = \langle H, H^t\rangle\langle t\rangle$ is a split extension, we only need to check that

(2) $$\langle H, H^t\rangle \cap G = H.$$

We have $U = H \cap \langle A, B\rangle = H^t \cap \langle A, B\rangle$ because this group is τ-invariant. Hence

$$\begin{array}{ccc} H & & H^t \\ & \diagdown\ \diagup & \\ & U & \end{array}$$

is a subamalgam of $\mathfrak{A}$ and (2) follows by the s.p.

Proof of Theorem 2. We need only to construct finite groups $K_1 = \langle G, t_1\rangle$ and $K_2 = \langle G, t_2\rangle$ such that, for all $H \subset G$ with $A \cap H = B \cap H = 1$, we have

(a) $\langle A, H\rangle \cap B = 1$ implies $\langle H, t_1\rangle \cap G = H$,

(b) $\langle B, H\rangle \cap A = 1$ implies $\langle H, t_2\rangle \cap G = H$.

Indeed, we can take $\langle G, t\rangle \hookrightarrow K_1 \times K_2$ with $g \to (g, g)$ for all $g \in G$ and $t \to (t_1, t_2)$ to prove the theorem.

We will construct $\langle G, t_1\rangle$ so that (a) is satisfied for some *given* $H \subset G$. Theorem 2 is then proved using a diagonal embedding into a finite direct product as above (with direct factors corresponding to various $H \subset G$). Thus, assume $\langle A, H\rangle \cap B = 1$.

We again begin with a copy G_1 of G and form an amalgam

$$\mathfrak{A} = \begin{tikzcd} \langle A_1, H_1\rangle & & G \\ & A_1 = B \arrow[ul, dash] \arrow[ur, dash] & \end{tikzcd}$$

recalling that $|A| = |B| = 2$. Let $M = \langle \mathfrak{A}\rangle$ be a finite group satisfying the s.p.

Next, form the amalgam

$$\mathfrak{B} = \begin{tikzcd} G_1 & & M \\ & \langle H_1, A_1\rangle \arrow[ul, dash] \arrow[ur, dash] & \end{tikzcd}$$

and let $N = \langle \mathfrak{B}\rangle$ be a finite group satisfying the s.p. Since

$$\begin{tikzcd} B_1 & & A \\ & 1 \arrow[ul, dash] \arrow[ur, dash] & \end{tikzcd}$$

is a subamalgam of $\mathfrak{B}$, we have

(3) $$\langle B_1, A\rangle \cap M = A \quad (\text{in } N),$$

and since

$$\begin{tikzcd} H_1 & & H \\ & 1 \arrow[ul, dash] \arrow[ur, dash] & \end{tikzcd}$$

is a subamalgam of $\mathfrak{A}$, we have

(4) $$\langle H_1, H\rangle \cap G = H \quad (\text{in } M),$$

and hence

(5) $$\langle H_1, H\rangle \cap A = 1 \quad (\text{in } M)$$

since $H \cap A = 1$.

Combining (3) and (5) gives

(6) $$\langle B_1, A\rangle \cap \langle H_1, H\rangle = 1 \quad (\text{in } N).$$

Let N' be a canonical copy of N. Since $D = \langle B_1, A\rangle$ and $D' = \langle B_1', A'\rangle$ are isomorphic by $B_1 \leftrightarrow A'$ and $A \leftrightarrow B_1'$, we can form an amalgam

$$\mathfrak{C} = \begin{array}{c} N \quad N' \\ \diagdown\diagup \\ D = D' \end{array}$$

in which $B_1 = A'$ and $A = B_1'$ and obtain a finite group $P = \langle \mathfrak{C}\rangle$ satisfying the s.p. Since

$$\begin{array}{c} \langle H_1, H\rangle \quad \langle H_1', H'\rangle \\ \diagdown\diagup \\ 1 \end{array}$$

is, by (6), a subamalgam of $\mathfrak{C}$, we have

$$\langle H, H_1, H', H_1'\rangle \cap N = \langle H, H_1\rangle \qquad (\text{in } P)$$

and from (4) it follows that

$$\langle H, H_1, H', H_1'\rangle \cap G = H \qquad (\text{in } P). \tag{7}$$

Our group P is a homomorphic image of the group Q with presentation

$$\langle G, G_1, G', G_1'\colon B = A_1, B_1 = A', B' = A_1', B_1' = A\rangle$$

and Q has an automorphism τ of order 4 which extends the canonical maps

$$G \xrightarrow{\tau} G_1 \xrightarrow{\tau} G' \xrightarrow{\tau} G_1' \xrightarrow{\tau} G \tag{8}$$

because all the relations in the above presentation of Q are preserved by these maps.

Put $P = Q/U$ and $\tilde{P} = Q/U \cap U^{\tau} \cap U^{\tau^2} \cap U^{\tau^3}$; so $\tilde{P} = \langle G, G_1, G', G_1'\rangle$ is finite and has an automorphism extending the maps (8), which we also call τ.

Let $\tilde{P}\langle t\rangle$ be a semidirect product where $|t| = 4$ and t induces τ on $\tilde{P}$. Note that relation (7) holds in $\tilde{P}$ also, because the natural homomorphism $f\colon \tilde{P} \to P$ is the identity on $G \cup G_1 \cup G' \cup G_1'$; so any violation of (7) in $\tilde{P}$ is preserved by f. Hence, it is clear that in $\langle G, t\rangle = \langle \tilde{P}, t\rangle$ we have $\langle H, t\rangle = \langle H, H_1, H', H_1'\rangle \langle t\rangle$ and (7) implies $\langle H, t\rangle \cap G = H$ as desired. Note that $A^t = A_1 = B$ because of (8) and the amalgamating relations of Q. This completes our proof.

The crucial requirement of this proof is to generate a finite group N from the amalgam

$$\begin{array}{c} G_1 \quad G \\ \diagdown\diagup \\ A_1 = B \end{array}$$

in such a way that

$\langle B_1, A\rangle \cap \langle H_1, H\rangle = 1$ in N. To do this using the subamalgam property, we made assumption (ii). If we try to relax the requirement that A and B have order 2, there is a further problem to guarantee that the subgroup $\langle B_1, A\rangle$ of N is symmetrically generated, so that the amalgam $\mathfrak{C}$ of the proof can be formed.

REFERENCES

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