COLLOQUIUM MATHEMATICUM

VOL. XV 1966 FASC. 1

SOME REMARKS ON INFINITESIMAL PROJECTIVE TRANSFORMATIONS IN RECURRENT AND RICCI-RECURRENT SPACES

BY

W. ROTER (WROCŁAW)

1. An n-dimensional ($n > 2$) Riemannian space is called *recurrent* [3], if the curvature tensor satisfies the condition

$$R_{hijk,l} = c_l R_{hijk} \neq 0 \tag{1}$$

for some vector c_j, where the comma indicates the covariant derivative.

Contracting (1) with g^{hk} we see that for a recurrent space the relation

$$R_{ij,l} = c_l R_{ij} \tag{2}$$

holds.

Spaces whose Ricci-tensor R_{ij} satisfies (2) for some vector c_j, where $n > 2$, are called *Ricci-recurrent*. We assume, moreover, that $R_{ij} \neq 0 \neq c_j$.

Thus every recurrent space with $R_{ij} \neq 0$ is Ricci-recurrent.

Let a Riemannian space admit an infinitesimal projective transformation with respect to the vector field v^j. Denote by $\mathscr{L}$ the Lie derivative with respect to this field. Then we have [4]:

$$\mathscr{L}\Gamma^i_{jk} = \delta^i_j A_k + \delta^i_k A_j, \tag{3}$$

$$\mathscr{L}R^h_{.ijk} = \delta^h_j A_{i,k} - \delta^h_k A_{i,j}, \tag{4}$$

$$\mathscr{L}R_{ij} = (1-n) A_{i,j}, \tag{5}$$

$$\mathscr{L}P^h_{.ijk} = 0, \tag{6}$$

where Γ^i_{jk} are the Christoffel symbols, A_j is a gradient vector field, and $P^h_{.ijk}$ denotes the projective curvature tensor, i. e.

$$P^h_{.ijk} = R^h_{.ijk} - \frac{1}{n-1}(\delta^h_k R_{ij} - \delta^h_j R_{ik}). \tag{7}$$

If $A_j = 0$, the infinitesimal projective transformation is an *affine* one.

Infinitesimal projective transformations in recurrent and Ricci-recurrent spaces, not necessarily of $\neq 0$ recurrent vector but with definite metric, have been studied by Prvanovitch [1].

For such recurrent and Ricci-recurrent spaces Prvanovitch proved:

(a) *If a Ricci-recurrent (recurrent) space admits an infinitesimal projection transformation, then it is an Einstein space or the vector field A_j satisfies the condition*

$$A_l R_{ij} R^{ij} + A_i R_{lj} R^{ij} + \tfrac{1}{2} R_{ij} R^{ij} \mathscr{L} c_l = 0.$$

(b) *If a compact Ricci-recurrent space, which is not an Einstein space, admits an infinitesimal projective transformation such that $\mathscr{L} c_j = 0$, then this transformation is a motion.*

In the present paper we shall investigate infinitesimal projective transformations in recurrent and Ricci-recurrent spaces, whose definitions have been given above.

Formulas

$$\mathscr{L} \Gamma^i_{jk} = \delta^i_j A_k + \delta^i_k A_j, \qquad A_j = -\tfrac{1}{3} \mathscr{L} c_j \tag{8}$$

(Theorem 1) as well as Theorems 2-4 are proved without assuming the definiteness of the metric. Further theorems are valid under the hypothesis that the metric is positive definite.

2. LEMMA 1. *If e_j and T_{ij} are numbers satisfying*

$$e_i T_{mj} + e_j T_{mi} = 0, \tag{9}$$

then either all the e_j are zero, or all the T_{ji} are zero.

Proof. Suppose that one of the e's, say e_q, is non-zero. Then (9) with $i = j = q$ gives $2 e_q T_{mq} = 0$, and therefore $T_{mq} = 0$ for all m.

Putting $i = q$ in (9) we now have $e_q T_{mj} = 0$, and therefore $T_{mj} = 0$ for all m and j.

LEMMA 2. *If the recurrent vector of a Ricci-recurrent space is a gradient, then the equation*

$$R_{ri} R^{r}_{\cdot j} = \tfrac{1}{2} R R_{ij} \tag{10}$$

holds, where $R = g^{ij} R_{ij}$.

Proof. It can be easily verified that, since c_j is a gradient, (2) gives

$$R_{ij,lm} - R_{ij,ml} = (c_{l,m} - c_{m,l}) R_{ij} = 0,$$

whence using Ricci's identity we obtain

$$R_{rj} R^{r}_{\cdot ilm} + R_{ir} R^{r}_{\cdot jlm} = 0. \tag{11}$$

By differentiating (11) covariantly and taking into consideration (2), we find

$$c_k(R_{rj}R^{r}_{\cdot ilm}+R_{ir}R^{r}_{\cdot jlm})+R_{rj}R^{r}_{\cdot ilm,k}+R_{ir}R^{r}_{\cdot jlm,k} = 0.$$

This, because of (11), yields

$$R^{r}_{\cdot j}R_{lmri,k}+R^{r}_{\cdot i}R_{lmrj,k} = 0. \tag{12}$$

Contracting (12) with g^{lk} and making use of the relation

$$R^{r}_{\cdot ijl,r} = R_{ij,l}-R_{il,j},$$

which follows easily from Bianchi's identity, we get

$$R^{r}_{\cdot j}(R_{mr,i}-R_{mi,r})+R^{r}_{\cdot i}(R_{mr,j}-R_{mj,r}) = 0.$$

But the last equation, in virtue of (2) and

$$c_r R^{r}_{\cdot j} = \tfrac{1}{2}Rc_j, \tag{13}$$

which is a immediate consequence of (2), and the formula $R^{r}_{\cdot j,r} = \frac{1}{2}R_{,j}$ (see [5], p. 19, equation (1.61)) is equivalent to

$$c_i(R_{rm}R^{r}_{\cdot j}-\tfrac{1}{2}RR_{mj})+c_j(R_{rm}R^{r}_{\cdot i}-\tfrac{1}{2}RR_{mi}) = 0. \tag{14}$$

Putting $T_{ij} = R_{ri}R^{r}_{\cdot j}-\frac{1}{2}RR_{ij}$ we see that (14) can be written as $c_iT_{mj}+c_jT_{mj} = 0$, and is therefore of the form (9).

Hence, since $c_j \neq 0$, Lemma 1 gives $T_{ij} = 0$, which proves our lemma.

LEMMA 3. *If the recurrent vector of a Ricci-recurrent space admitting an infinitesimal projective transformation is a gradient, then the Ricci-tensor satisfies the equation*

$$A_{r,k}R^{r}_{\cdot l} = A_{r,l}R^{r}_{\cdot k}. \tag{15}$$

Proof. Applying to (11) the Lie derivative and making use of (4) and (5), we find

$$(1-n)A_{i,r}R^{r}_{\cdot jlm}+A_{j,m}R_{il}-A_{j,l}R_{im}+A_{i,m}R_{lj}-A_{i,l}R_{mj}+ \\ +(1-n)A_{r,j}R^{r}_{\cdot ilm} = 0.$$

The contraction of the last equation with g^{ij} gives

$$(1-n)A_{r,s}R^{rs}_{\cdot\cdot lm}+A_{r,m}R^{r}_{\cdot l}-A_{r,l}R^{r}_{\cdot m}+A_{r,m}R^{r}_{\cdot l}-A_{r,l}R^{r}_{\cdot m}+ \\ +(1-n)A_{r,s}R^{rs}_{\cdot\cdot lm} = 0.$$

Whence, since $A_{i,j}$ is symmetric and, consequently, $A_{r,s}R^{rs}_{\cdot\cdot lm} = 0$, we obtain (15).

LEMMA 4. *If the scalar curvature of a Ricci-recurrent space admitting infinitesimal projective transformation is* $\neq 0$ *then the vector field* A_j *satisfies the condition*

$$A^{r}_{\cdot,r} = 0. \tag{16}$$

Proof. M. Prvanovitch ([1], equations (2.11) and (2.12)) proved that if the Ricci-recurrent space, whose recurrent vector is a gradient, admits an infinitesimal projective transformation, then the following equations hold:

$$A^{r}_{\cdot,r}R^{st}R_{st} - RA_{r,s}R^{rs} = 0, \tag{17}$$

$$RA^{r}_{\cdot,r} - nA_{r,s}R^{rs} = 0. \tag{18}$$

Contracting (2) with g^{ij}, we obtain $R_{,l} = c_l R$, whence it follows that c_j is a gradient. Hence, the equations (10) and (15) are satisfied.

It is easy to see that (10) gives

$$R^{rs}R_{rs} = \tfrac{1}{2}R^2. \tag{19}$$

Substituting (19) into (17), we get $\frac{1}{2}R^2A^{r}_{\cdot,r} - RA_{r,s}R^{rs} = 0$. This, together with (18), yields $(n-2)RA_{r,s}R^{rs} = 0$. But the last equation, in virtue of (18), implies $R^2A^{r}_{\cdot,r} = 0$, which proves our lemma.

LEMMA 5. *If the Ricci-recurrent space with* $R \neq 0$ *admits an infinitesimal projective transformation, then its Ricci-tensor satisfies the equation*

$$(\mathcal{L}c_r)R^{r}_{\cdot j} = -3A_rR^{r}_{\cdot j}. \tag{20}$$

Proof. Applying to (2) the Lie derivative and using (5), (3), and the well-known formula

$$\mathcal{L}T_{ij,l} = (\mathcal{L}T_{ij})_{,l} - T_{rj}\mathcal{L}\Gamma^{r}_{il} - T_{ir}\mathcal{L}\Gamma^{r}_{jl},$$

we get

$$(1-n)A_{i,jl} - A_iR_{lj} - 2A_lR_{ij} - A_jR_{il} = (\mathcal{L}c_l)R_{ij} + (1-n)c_lA_{i,j}. \tag{21}$$

The contraction of (21) with g^{ij}, in virtue of (16), yields

$$R\mathcal{L}c_l = -2RA_l - 2A_rR^{r}_{\cdot l}. \tag{22}$$

Contracting further this with $R^{l}_{\cdot k}$ and taking into consideration (10), we obtain easily (20).

LEMMA 6. *If the Ricci-recurrent space with* $R \neq 0$ *admits an infinitesimal projective transformation, then the vector field* A_j *satisfies the following equation*:

$$A_rR^{r}_{\cdot j} = \tfrac{1}{2}RA_j. \tag{23}$$

Proof. Contracting (21) with g^{il} and using (20), we obtain

$$(1-n)(A^{r}_{.,jr}-c_r A^{r}_{.,j}) = RA_j.$$

Since $A^{r}_{.,jr} = A^{r}_{.,rj} + A_r R^{r}_{.j}$, which easily follows from Ricci's identity, the last but one equation, in view of (16), can be written as

$$(24)\qquad (1-n)(A_r R^{r}_{.j} - c_r A^{r}_{.,j}) = RA_j.$$

This, by contraction with $R^{j}_{.k}$, yields

$$(1-n)(A_r R^{r}_{.j} R^{j}_{.k} - c^r A_{r,j} R^{j}_{.k}) = RA_j R^{j}_{.k},$$

or, because of (15),

$$(1-n)(A_r R^{r}_{.j} R^{j}_{.k} - c_r R^{rj} A_{j,k}) = RA_j R^{j}_{.k}.$$

Making now use of (10) and (13), we find

$$\tfrac{1}{2}(1-n)R(A_r R^{r}_{.k} - c_r A^{r}_{.,k}) = RA_r R^{r}_{.k}.$$

But this together with (24) implies $\frac{1}{2}R^2 A_k = RA_r R^{r}_{.k}$, which gives (23). Our lemma is thus proved.

3. Substituting now (23) into (22), we have $R\mathscr{L}c_l = -3RA_l$. But this, when $R \neq 0$, gives

$$(25)\qquad \mathscr{L}c_j = -3A_j.$$

Hence, we have the following

THEOREM 1. *If the Ricci-recurrent space with $R \neq 0$ admits an infinitesimal projective transformation, then this transformation satisfies* (8).

THEOREM 2. *If the Ricci-recurrent space with $R \neq 0$ admits an infinitesimal projective transformation, then this transformation is an affine one if and only if $\mathscr{L}R_{ij} = 0$.*

Proof. If $A_j = 0$, then the equation $\mathscr{L}R_{ij} = 0$ follows immediately from (5).

Suppose now $\mathscr{L}R_{ij} = 0$. Then, in virtue of (5), we have $A_{i,jk} = 0$, which, using the Ricci identity, gives

$$(26)\qquad A_{i,jk} - A_{i,kj} = -A_r R^{r}_{.ijk} = 0.$$

Contracting (26) with g^{ij} and taking into consideration (23), we obtain $\frac{1}{2}RA_k = 0$. This, since $R \neq 0$, completes the proof.

THEOREM 3. *If the Ricci-recurrent space with $R \neq 0$ admits an infinitesimal projective transformation such that $(\mathscr{L}c_j)_{,k} = 0$, then this transformation is an affine one.*

This result follows easily from (25) and Theorem 2.

THEOREM 4. *If a recurrent space with $R \neq 0$ admits an infinitesimal projective transformation, then this transformation is always an affine one.*

Proof. M. Prvanovitch ([1], equation (2.7), p. 220) proved that the projective curvature tensor of a recurrent space, admitting an infinitesimal projective transformation, satisfies the relation

$$(\mathcal{L}c_l)P_{tijk} = g_{tl}A_r P^{r}_{\cdot ijk} - 2A_l P_{tijk} - A_i P_{tljk} - A_j P_{tilk} - A_k P_{tijl}.$$

Because of $P^{r}_{\cdot rij} = P^{r}_{\cdot irj} = P^{r}_{\cdot ijr} = 0$, the contraction of the last equation with g^{tl} gives

$$(27) \qquad (\mathcal{L}c_r)P^{r}_{\cdot ijk} = (n-2)A_r P^{r}_{\cdot ijk}.$$

Substituting (7) into (27) and contracting with g^{ij}, we obtain

$$n(\mathcal{L}c_r)R^{r}_{\cdot k} - R\mathcal{L}c_k = (n-2)(nA_r R^{r}_{\cdot k} - RA_k).$$

Making now use of (20), (23) and (25), we obtain easily $(n+1)RA_k = 0$, which proves our theorem.

4. Suppose now that the metric of the investigated space is positive definite. Then, contracting (2) with R^{ij}, we get

$$(R^{ij}R_{ij})_{,l} = 2c_l R^{ij}R_{ij},$$

whence, since $R^{ij}R_{ij} \neq 0$, it follows that c_j is a gradient.

Taking into consideration (19), which now holds, it can be easily verified that the scalar curvature is $\neq 0$. Therefore the equations (16), (23) and (25) are satisfied.

Hence, we have

THEOREM 5. *If the Ricci-recurrent space with positive definite metric admits an infinitesimal projective transformation, then this transformation satisfies* (8).

Since the scalar curvature of a recurrent space with positive definite metric cannot be zero, which is an immediate consequence of the relation $R^{hijk}R_{hijk} = R^2$ (see [2], equation (10)), Theorem 4 yields

THEOREM 6. *If a recurrent space with positive definite metric admits an infinitesimal projective transformation, then this transformation is always an affine one.*

Now, the following theorem will be proved

THEOREM 7. *If a compact orientable Ricci-recurrent space with positive definite metric admits an infinitesimal projective transformation, then this transformation is a motion.*

Proof. Applying to the inner product $v^r c_r$ the Laplace operator Δ, we obtain

$$\Delta(v^r c_r) = g^{ij}(v^{r}_{,i}c_r + v^r c_{r,i})_{,j},$$

which, since c_j is a gradient, can be written in the form

$$\Delta(v^r c_r) = g^{ij}(v^r_{.,i} c_r + v^r c_{i,r})_{,j}.$$

Taking into account the well-known formula $\mathscr{L} a_j = v^r a_{j,r} + v^r_{.,j} a_r$, we obtain in our case $\Delta(v^r c_r) = g^{ij}(\mathscr{L} c_i)_{,j}$, or, in virtue of (25), $\Delta(v^r c_r) = -3A^r_{.,r}$.

But, since the metric is positive definite, equation (16) holds, and therefore $\Delta(v^r c_r) = 0$ everywhere in the space.

Making now use of Bochner's theorem ([5], p. 30), we obtain $v^r c_r = \mathrm{const}$ and $\mathscr{L} c_j = (v^r c_r)_{,j} = 0$.

Hence, because of (25), this transformation is an affine one. But it is known ([5], p. 58) that an infinitesimal affine transformation in a compact orientable Riemannian space is always a motion. This remark completes the proof of our theorem.

The following theorem is a consequence of Theorem 6:

THEOREM 8. *If a compact orientable recurrent space with positive definite metric admits an infinitesimal projective transformation, then this transformation is a motion.*

REFERENCES

[1] M. Prvanovitch, *Projective and conformal transformations in recurrent and Ricci-recurrent Riemannian spaces*, Tensor 12,3 (1962), p. 219-226.

[2] W. Roter, *Quelques remarques sur les espaces récurrents et Ricci-récurrents*, Bulletin de l'Académie Polonaise des Sciences, Série des Sciences Mathématiques, Astronomiques et Physiques, 10 (1962), p. 533-536.

[3] H. S. Ruse, *Three-dimensional spaces of recurrent curvature*, Proceedings of the London Mathematical Society 50 (1949), p. 438-446.

[4] K. Yano, *The theory of Lie derivatives and its applications*, Amsterdam 1957.

[5] — and S. Bochner, *Curvature and Betti numbers*, Princeton 1953.

INSTITUTE OF MATHEMATICS OF THE WROCŁAW UNIVERSITY

Reçu par la Rédaction le 20. 1. 1965