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# A PROPER CLASS OF PAIRWISE INCOMPARABLE CARDINALS

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**1. Introduction.** There are a lot of well-known consequences of the axiom of choice (AC). One of them is the linear ordering of the class of cardinals. Actually it is equivalent to AC. Therefore, if AC is false, then there are at least two incomparable cardinals. It is possible to obtain a model with arbitrarily large set of pairwise incomparable cardinals via the Embedding Theorem. We ask if the existence of a proper class of pairwise incomparable cardinals is relatively consistent with ZF.

Another and more general question is how much, without AC, the partial ordering of cardinals can be different from the well-ordering of alephs. Jech shows in [2] that every partially ordered set can be embedded into cardinals.

The Theorem below is a generalization of the results mentioned above and answers our questions.

THEOREM. *Let $M$ be a countable standard model (c.s.m.) for* ZFC. *Let $\langle I, \leqslant\rangle$ be a partial ordering, $I$ and $\leqslant$ be classes in $M$, and let every initial segment of $\langle I, \leqslant\rangle$ belong to $M$. Then there exist a c.s.m. $N$ for* ZFC, *which is a symmetric extension of $M$, and a class $S = \{S_i\colon i \in I\}$ in $N$ such that, for $i, j \in I$,*

$$i \leqslant j \leftrightarrow N \models |S_i| \leqslant |S_j|.$$

From this we obtain the following

COROLLARY.

Cons ZF $\leftrightarrow$ Cons(ZF + "*there is a proper class of pairwise incomparable cardinals*").

The Embedding Theorem cannot be applied to our proof. To replace sets with classes we use a forcing with proper classes. To destroy the axiom of choice we need symmetric methods. For details see [1]–[4].

**2. Construction.** Let $M$ and $\langle I, \leqslant\rangle$ satisfy the assumptions of the Theorem, i.e., $M$ is a c.s.m. for ZFC (in the language extended with two additional predicates $I(x)$ and $\leqslant(x, y)$ interpreted as $x \in I$ and $x \leqslant y$). The set $I$ is partially ordered by $\leqslant$, and every initial segment of $\langle I, \leqslant\rangle$ is an element of $M$.

Without loss of generality we may assume that $I$ is a proper class in $M$ and, moreover, there is a well-ordering $W$ of $M$ which is a class in $M$. One can add $W$ by the use of a set-closed notion of forcing. Since $\langle I, \leqslant\rangle$ can usually be embedded into $\langle \mathscr{P}(I)\cap M, \subseteq\rangle$, we will deal with $\langle \mathscr{P}(\mathrm{On}^M)\cap M, \subseteq\rangle$.

The first step is the use of an Easton-like notion of forcing. It will "add" $\omega$ new subsets to every $\omega_{\alpha+1}$ for $\alpha\in\mathrm{On}$.

DEFINITION 1.

$$p\in P_\sigma \leftrightarrow \mathrm{func}(p)\,\&\,(\alpha,\beta,m)(\langle\alpha,\beta,m\rangle\in\mathrm{dom}(p)\to\alpha\leqslant\delta\ \&\ \beta<\omega_\alpha\ \&\ m<\omega)$$
$$\&\,(\mathrm{rng}(p)\subseteq 2)\,\&\,(\gamma)_\delta(\omega_\gamma \text{ is regular } \to |p\restriction(\gamma+1)\times\omega_\gamma\times\omega| < \omega_\gamma),$$

and $\leqslant$ is the reversed inclusion.

Let us note that $P$ is an Easton-like notion of forcing. Let $K$ be $P$-generic over $M$. Then $M[K]\models\mathrm{ZFC}$.

In the second step we define a symmetric submodel $N$ of $M[K]$ satisfying all the requirements of the Theorem. First we define

(a) a sequence of groups of automorphisms of $P_\alpha$'s;

(b) a sequence of normal filters of subgroups of the groups defined in (a).

DEFINITION 2.

$$\pi\in G_\alpha \leftrightarrow \pi = \langle \pi_\beta\colon\ \beta\leqslant\alpha,\ \pi_\beta \text{ is a permutation of } \omega\rangle,$$

for $p\in P$ and $\pi\in G_\alpha$,

$$\pi p = \{\langle\langle\beta,\gamma,\pi_\beta(m)\rangle, i\rangle\colon\ \langle\langle\beta,\gamma,m\rangle, i\rangle\in p\}.$$

One can check that $G_\alpha$ is a group of automorphisms of $P_\alpha$.

DEFINITION 3.

$$H_e = \{\pi\in G_\alpha\colon\ \pi_\beta(m) = m \text{ for } (\beta, m)\in e\}.$$

$F_\alpha$ is the filter generated by

$$\{H_e\colon\ e \text{ is a finite subset of } (\alpha+1)\times\omega\}.$$

FACT 4. *For every* $\alpha\in\mathrm{On}$, $F_\alpha$ *is normal.*
*The sequences* $\langle G_\alpha\colon\ \alpha\in\mathrm{On}\rangle$, $\langle F_\alpha\colon\ \alpha\in\mathrm{On}\rangle$ *are coherent, i.e.,*

$$(\alpha)(\beta)_\alpha(G_\beta = \{\pi\restriction P_\beta\colon\ \pi\in G_\alpha\});$$
$$(\alpha)(\beta)_\alpha(F_\beta = \{\{\pi\restriction P_\beta\colon\ \pi\in H\}\colon\ H\in F_\alpha\}).$$

DEFINITION 5. $N\models \bigcup_{\alpha\in\mathrm{On}} M[K_\alpha/F_\alpha]$.

FACT 6. $N\models\mathrm{ZF}$.

**3. Properties of** $N$. Now we define some auxiliary sets and their names:

$$x_{\alpha,n} = \{\beta\in\omega_{\alpha+1}\colon\ (\bigcup K)(\langle\alpha+1,\beta,n\rangle) = 0\} \quad \text{for } n<\omega,\ \alpha\in\mathrm{On};$$

$$\mathrm{dom}(\mathbf{x}_{\alpha,n}) = \{\beta^{\vee}:\ \beta\in\omega_{\alpha+1}\},$$

$$\mathbf{x}_{\alpha,n}(\beta^{\vee}) = \{\langle\langle\alpha+1,\beta,n\rangle,0\rangle\}.$$

For $i\subseteq \mathrm{On}$:

$$\mathbf{S}_i = \{\mathbf{x}_{\alpha,n}:\ \alpha\in i,\ n<\omega\},\qquad \mathrm{dom}(\mathbf{S}_i) = \{\mathbf{x}_{\alpha,n}:\ \alpha\in i,\ n<\omega\};$$

$$\mathbf{S}_i(\mathbf{x}_{\alpha,n}) = 1_P.$$

FACT 7. *The names $\mathbf{x}_{\alpha,n}$ and $\mathbf{S}_i$ are symmetric.*

THE MAIN LEMMA. $i\subseteq j \leftrightarrow |S_i|\leqslant |S_j|$.

Proof. $\rightarrow$ is obvious. To show $\leftarrow$ let us assume that $|S_i|\leqslant|S_j|$ and $\neg(i\subseteq j)$. Then there exists $\alpha_0$ such that $\alpha_0\in i$ and $\alpha_0\notin j$. Let $f$: $S_i\to S_j$ and let $\mathbf{f}$ be a symmetric name for the function $f$. Then for some $\delta$ we have $\mathbf{f}\in(M)^{P_\delta}$. There is $p_0\in K_\delta$ such that

$$p_0 \Vdash \text{“}\mathbf{f} \text{ is a function from } \mathbf{S}_i \text{ to } \mathbf{S}_j\text{”}.$$

To get a contradiction we will find $q\leqslant p_0$ such that

$$q\Vdash \text{“}\mathbf{f} \text{ is not 1-1”}.$$

Let $e$ be a finite subset of $(\alpha+1)\times\omega$ and let

$$\mathrm{sym}(\mathbf{f})\supseteq H_e.$$

Then there exist $p\leqslant p_0$, $\beta_0\neq\alpha_0$, $n<\omega$ and $l<\omega$ such that

$$\langle\alpha_0,n\rangle\notin e\quad\text{and}\quad p\Vdash \mathbf{f}(\mathbf{x}_{\alpha_0,n}) = \mathbf{x}_{\beta_0,l}.$$

We want to construct $\pi\in G_\delta$ satisfying the following conditions:

(i) $p$ and $\pi p$ are compatible;
(ii) $\pi\in H_e$;
(iii) $\pi_\beta = \mathrm{id}$ for $\beta\neq\alpha_0$;
(iv) $\pi_{\alpha_0}(n)\neq n$.

By (ii), $\pi\mathbf{f}=\mathbf{f}$, and by (iii) it follows that

$$\pi\mathbf{x}_{\beta_0,l} = \mathbf{x}_{\beta_0,l}\quad\text{and}\quad p\Vdash(\pi\mathbf{f})(\pi\mathbf{x}_{\alpha_0,n}) = \pi\mathbf{x}_{\beta_0,l}.$$

In view of (i) there is $q$ with $q\leqslant\pi p$ and $q\leqslant p$ such that

$$q\Vdash \mathbf{f}(\mathbf{x}_{\alpha_0,n}) = \mathbf{x}_{\beta_0,l}\ \&\ \mathbf{f}(\mathbf{x}_{\alpha_0},\pi_{\alpha_0}(n)) = \mathbf{x}_{\beta_0,l}.$$

To get $\pi$ let $m\in\omega$ and

$$\langle\alpha_0,m\rangle\notin e\quad\text{and}\quad\langle\alpha_0+1,\gamma,m\rangle\notin\mathrm{dom}(p)$$

for any $\gamma<\omega_{\alpha_0+1}$. Let $\pi_{\alpha_0}$ replace $m$ and $n$ and assume that $\pi$ does not move anything else. Then $\pi$ satisfies (i)–(iv), and therefore the proof is completed.

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