COLLOQUIUM MATHEMATICUM

VOL. XXX 1974 FASC. 1

AGASSIZ SUM OF ALGEBRAS

BY

G. GRÄTZER AND J. SICHLER (WINNIPEG, MANITOBA)

A colimit-type construction of an algebra over a system of algebras without nullary operations indexed by a semilattice was introduced by Płonka in [2]. Lakser, Padmanabhan and Platt generalized it to the concept of Płonka sum described in [1]; Płonka sum applies also to algebras having nullaries. Both constructions are very natural and have already found numerous applications.

In the present note* we would like to generalize these concepts still further. The new concept of Agassiz sum does not impose any restrictions on the indexing algebra and a corresponding extension of the principal result of [2] remains valid also in the case of Agassiz sums.

Let K be a class of algebras of type τ, and let I (I for Indexing Algebras) be a class of algebras of type ϱ. The only assumption we make is that algebras of I have a nullary operation whenever algebras of K do.

To every polynomial symbol $\boldsymbol{p}$ of type τ we assign a polynomial symbol $N(\boldsymbol{p})$ of type ϱ (N for the Name of the Polynomial) satisfying

(i) the variables of $\boldsymbol{p}$ and $N(\boldsymbol{p})$ are the same;

(ii) N preserves composition, that is,

$$N\big(\boldsymbol{p}(\boldsymbol{q}_1, \ldots, \boldsymbol{q}_k)\big) = N(\boldsymbol{p})\big(N(\boldsymbol{q}_1), \ldots, N(\boldsymbol{q}_k)\big)$$

is an identity in I.

These two conditions say that N is a product-preserving functor from the theory of K into the theory of I.

Let K, I and N be given and let $\boldsymbol{B}$ be an algebra of the indexing class I. Let $R \subseteq B^2$ be a transitive relation on the underlying set B of the algebra $\boldsymbol{B}$ such that

(a) if $j = h(b_1, \ldots, i, \ldots, b_n)$ for some algebraic operation h of $\boldsymbol{B}$ and for some $b_1, \ldots, b_n \in B$, then $\langle i, j\rangle \in R$.

* The work of both authors was supported by the National Research Council of Canada.

An *Agassiz system* $\boldsymbol{S}$ of algebras over $\boldsymbol{B}$ is a family $(\boldsymbol{A}_i\colon i \in B)$ of algebras from K together with a family $(f_{ij}\colon \langle i, j\rangle \in R)$ of homomorphisms $f_{ij}\colon \boldsymbol{A}_i \to \boldsymbol{A}_j$ such that

(b) $f_{jk} \circ f_{ij} = f_{ik}$ whenever $\langle i, j\rangle, \langle j, k\rangle \in R$.

Form the disjoint union $A = \bigcup (A_i\colon i \in B)$ and define an algebra of type τ on A as follows. Let $\boldsymbol{p}$ be an n-ary polynomial symbol and $n \geqslant 1$. For $a_1, \ldots, a_n \in A$, let $b_1, \ldots, b_n$ be the (uniquely determined) elements of B such that $a_i \in A_{b_i}$, and let $b = N(\boldsymbol{p})(b_1, \ldots, b_n)$. By (a), $\langle b_i, b\rangle \in R$, so we can set $a_i^* = f_{b_i,b}(a_i)$. All a_i^* are in A_b; we write

$$p(a_1, \ldots, a_n) = p_{A_b}(a_1^*, \ldots, a_n^*).$$

If $\boldsymbol{p}$ is nullary, then $N(\boldsymbol{p})$ is nullary and we define p on A to be the value of $p_{A_{N(\boldsymbol{p})}}$. Let $\boldsymbol{A}$ be the algebra on the set A whose operations have just been defined. The algebra $\boldsymbol{A}$ is called the *Agassiz sum* of $\boldsymbol{S}$ and denoted by $\boldsymbol{A} = \lim_N(\boldsymbol{S})$.

Example 1. Let the algebras of K have no nullary operations and let I be the class of semilattices. For a polynomial symbol $\boldsymbol{p}$, set

$$N(\boldsymbol{p}) = \big((\boldsymbol{x}_1 \vee \boldsymbol{x}_2) \vee \ldots\big) \vee \boldsymbol{x}_k,$$

where $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_k$ are all the variables of $\boldsymbol{p}$. If the relation R is the partial ordering of a semilattice $\boldsymbol{B}$, then the Agassiz sum of the corresponding Agassiz system is the sum described in [2].

Example 2. If nullary operations are permitted to appear in the algebras of Example 1, the Płonka sum of [1] is obtained.

Example 3. The direct product $\boldsymbol{A} \times \boldsymbol{B}$ of two algebras of the same type is obtained as the Agassiz sum with the naming functor $N(\boldsymbol{p}) = \boldsymbol{p}$ of an Agassiz system consisting of $|\boldsymbol{B}|$ copies of the algebra $\boldsymbol{A}$ and of all the canonical isomorphisms between them.

A large variety of examples can be constructed from

Proposition. *Let I be the class of semigroups and let I_0 be the class of semigroups with* 0. *For a polynomial symbol $\boldsymbol{p}$ of type τ, write $N(\boldsymbol{p}) = \big((\boldsymbol{x}_1 \cdot \boldsymbol{x}_2) \cdot \ldots\big) \cdot \boldsymbol{x}_k$, where $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_k$ lists all variables of $\boldsymbol{p}$ (with repetition) in the order of their occurrence. If $\boldsymbol{p}$ is nullary, set $N(\boldsymbol{p}) = 0$. Then N satisfies* (i) *and* (ii) *for any class K.*

Let $\lim_N(K, I)$ denote the class of all isomorphic copies of all Agassiz sums with given K, I and N. Let $\mathrm{Id}(K)$ be the set of all identities that hold in K. An identity $\boldsymbol{p} = \boldsymbol{q}$ in $\mathrm{Id}(K)$ is *N-regular* if $N(\boldsymbol{p}) = N(\boldsymbol{q})$ holds in I. Let $\mathrm{Id}_N(K)$ be the set all N-regular identities.

Theorem. $\mathrm{Id}\big(\lim_N(K, I)\big) = \mathrm{Id}_N(K)$.

If this theorem is specialized to the case described in Example 1, it becomes the main result of [2]. A result of [1] is obtained if that theorem is applied to Example 2.

Observe that $\mathrm{Id}_N(K) = \mathrm{Id}(K) \cap \mathrm{Id}(I) = \mathrm{Id}(K \cup I)$ whenever K and I are of the same type and the functor N is trivial. Let K and I be equational classes of the same type. It is natural to ask under what conditions can every algebra of $K \vee I$ be represented as an Agassiz sum. S. M. Lee has shown that this happens in several cases of pairs of equational classes of idempotent semigroups.

PROBLEM 1. Let K and L be equational classes of algebras of the same type and let $K \subseteq L$. What conditions are sufficient for the existence of an equational class I and a naming functor N with $L = \lim_N(K, I)$? (**P 892**)

An identity $p = q$ is *regular* if the same variables occur on both its sides. Thus, in Example 1 we always get $\mathrm{Id}(\lim_N(K, I))$ as a well-defined subset of $\mathrm{Id}(K)$.

PROBLEM 2. Under what conditions can $\Sigma \subseteq \mathrm{Id}(K)$ be represented in the form $\Sigma = \mathrm{Id}(\lim_N(K, I))$ for some I and N? (**P 893**)

REFERENCES

[1] H. Lakser, R. Padmanabhan and C. R. Platt, *Equational classes defined by regular identities* (to appear).

[2] J. Płonka, *On a method of construction of abstract algebras*, Fundamenta Mathematicae 61 (1967), p. 183-189.

UNIVERSITY OF MANITOBA
WINNIPEG, MANITOBA, CANADA

Reçu par la Rédaction le 28. 3. 1973