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REMARKS ON SUPEREXTENSIONS

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Following de Groot [1], a topological space K is called *supercompact* if it has an open subbase B such that each covering of K by means of B contains a two-element subcovering. The subbase B is called a *supersubbase*. For a given topological space X and for a given closed subbase S on X there exists ([1], see also [6] for details) a supercompact space $\lambda_S X$, called a *superextension of X with respect to S*, which contains X, not necessarily as a dense subset. If S consists of all closed subsets of X, then $\lambda_S X$ is denoted by λX. Van Mill [5] distinguished a class of regular supercompact spaces K which, *inter alia*, have the following property: K is the superextension of each of its dense subsets.

We shall prove (Theorem 1) that for each regular supercompact space K with density m there exists a regular supercompact compactification rm of an infinite discrete space m of cardinality m such that $K = rm\setminus m$. Van Mill [5] proved this theorem in the case where K is a product of ordered spaces (obviously, supercompact).

We shall also prove that for each supercompact space K there exists a closed subbase S on a discrete space m, $\operatorname{card} m = \operatorname{card} K$, such that K is homeomorphic to the space of all free maximal linked subfamilies relative to the subbase S.

At the end we shall prove that each supercompact space which has a special subbase, called a *normal binary closed subbase*, is a continuous image of $\lambda m\setminus m$, where m is a discrete space with $\operatorname{card} m = \operatorname{card} K$.

1. Preliminaries. Let X be a topological space. A subbase S for closed subsets of X is called:

binary if for each subfamily S' of S such that $\bigcap S' = \emptyset$ there exist U and V in S' such that $U\cap V = \emptyset$;

a *T_1-subbase* if for each $x\in X$ and for each $V\in S$ such that $x\notin V$ there exists a $U\in S$ such that $x\in U$ and $U\cap V = \emptyset$;

normal if for each V, $T \in S$ such that $V \cap T = \emptyset$ there exist V' and T' in S such that $V \subset V'$, $T \subset T'$, $T' \cap V = \emptyset = T \cap V'$ and $T' \cup V' = X$.

A subfamily M of S is called *linked* if every two of its members meet. A linked subfamily M of S is called *fixed* if $\bigcap M \neq \emptyset$, and *free* if $\bigcap M = \emptyset$. The Kuratowski-Zorn lemma implies that every linked subfamily is contained in a maximal one.

Let

$$\lambda_S X = \{M \subset S \colon M \text{ is a maximal linked subfamily in } S\}.$$

For $A \subset X$ we set

$$A^+ = \{M \colon M \in \lambda_S X \text{ and there exists } V \in M \text{ such that } V \subset A\}.$$

We take the family of sets V^+, where $V \in S$ as a closed subbase for a topology on $\lambda_S X$; with this topology, $\lambda_S X$ is called a *superextension of X relative to the subbase S*. In the case where S consists of all closed subsets of X, $\lambda_S X$ is denoted by λX and is called, shortly, the *superextension of X*. All these notions were introduced by de Groot and can be found in Verbeek's book [6].

PROPOSITION (see [6]). (a) *If S is a T_1-subbase, then X is embeddable in $\lambda_S X$.*

(b) *If S is a normal T_1-subbase, then $\lambda_S X$ is T_2.*

(c) *$\lambda_S X$ is supercompact; the supersubbase is* $\{(X \setminus V)^+ \colon V \in S\}$.

THEOREM (van Mill [5]). Let $X \subset Y$. *Then Y is homeomorphic to a superextension of X if and only if Y has a binary closed subbase S such that if V, $T \in S$ and $V \cap T \neq \emptyset$, then $V \cap T \cap X \neq \emptyset$.*

THEOREM (Jensen [2]). *Let S be a T_1-subbase for X. Let T be a normal T_1-subbase for Y and let $f \colon X \to Y$ be continuous and such that $f^{-1}(V) \in S$ for each $V \in T$. Then f can be extended to a continuous map $\bar{f} \colon \lambda_S X \to \lambda_T Y$. Moreover, if f is onto, then so is $\bar{f}$.*

If A is a family of subsets of a set X, then the minimal family containing A and closed with respect to the finite union and the finite intersection will be called the *ring generated by A*.

A space X is called *regular supercompact* if it has a binary closed subbase S such that the ring generated by S consists of regular closed sets.

Let m be an infinite discrete space of cardinality m and let K be a compact space with a dense subset M of cardinality less than or equal to m.

The set rm will be defined as a subset of the disjoint union of K and of the subset $Y \doteq M \times m$ of the product $K \times m$.

Let T be a given open subbase of K. Generate the topology on rm by the subbase S consisting of

(1) the sets $U \cup (U \cap M \times m)$, where $U \in T$;
(2) the one-point sets from Y;
(3) complements of one-point sets in Y.

2. Theorems.

LEMMA 1 (A. K. Steiner and E. F. Steiner [4]). *Let m be an infinite discrete space and let K be a compact Hausdorff space with a dense subset of cardinality less than or equal to that of m. Then m has a compactification rm with $K = rm \setminus m$.*

Note. Steiners obtained that lemma by taking "graph-closure" compactification. The method used here is a generalization of the Alexandroff construction for the double circumference (cf. [3]).

Proof. Let M be a dense subset of K such that $\operatorname{card} M \leqslant m$.

We shall prove that rm is compact, Hausdorff, and that Y is a dense subset of rm.

To prove that Y is dense, let $U_1, \ldots, U_k$ be a finite family of elements from S such that $U_1 \cap \ldots \cap U_k \neq \emptyset$. If one of these elements is of the form $\{y\}$, where $y \in Y$, then

$$U_1 \cap \ldots \cap U_k = \{y\}.$$

If these elements are of the form $rm \setminus \{y\}$, where $y \in Y$, then

$$U_1 \cap \ldots \cap U_k \cap Y \neq \emptyset.$$

If $U_i = V_i \cup (V_i \cap M \times m)$ for $i = 1, \ldots, n$, and $U_j = rm \setminus \{y\}$ for $j = n+1, \ldots, k$, then

$$\begin{aligned} U_1 \cap \ldots \cap U_k &= [V_1 \cup (V_1 \cap M \times m)] \cap \ldots \cap [V_n \cup (V_n \cap M \times m)] \cap \\ &\quad \cap (rm \setminus \{y_{n+1}\}) \cap \ldots \cap (rm \setminus \{y_k\}) \\ &= (V_1 \cap \ldots \cap V_n) \cup ((V_1 \cap \ldots \cap V_n) \cap M \times m) \setminus \{y_{n+1}, \ldots, y_k\}. \end{aligned}$$

Since M is dense in K, the set $V_1 \cap \ldots \cap V_n \cap M \times m$ is non-empty and infinite. Hence $(V_1 \cap \ldots \cap V_n \cap M \times m) \setminus \{y_{n+1}, \ldots, y_k\}$ is non-empty.

To prove that rm is compact, let $R \subset S$ be an open covering of rm, i.e., $\bigcup R = rm$. If $rm \setminus \{y\} \in R$, then there exists an element $U \in R$ such that $y \in U$. Hence $(rm \setminus \{y\}) \cup U = rm$. If no element $rm \setminus \{y\}$ belongs to R, then

$$K = \bigcup \{U \colon U \cup (U \cap M \times m) \in R\}.$$

Since K is compact, there exists a finite subfamily $R' \subset R$ such that

$$K = \bigcup \{U \colon U \cup (U \cap M \times m) \in R'\}.$$

Hence $rm = \bigcup R'$.

To prove that rm is Hausdorff, let y and z be two different points from rm. If $y, z \in Y$, then y and z are open. If $z \in K$ and $y \in Y$, then

$$z \in rm \setminus \{y\} \quad \text{and} \quad \{y\} \cap (rm \setminus \{y\}) = \emptyset.$$

If $y, z \in K$, then there exist U and V open in K such that $y \in U$, $z \in V$ and $U \cap V = \emptyset$. Hence $U \cup (U \cap M \times m)$ and $V \cup (V \cap M \times m)$ are disjoint neighbourhoods of y and z in rm.

LEMMA 2. *Let m be an infinite discrete space and let K be a supercompact space with a dense subset of cardinality less than or equal to that of m. Then rm constructed in Lemma* 1 *is supercompact.*

Proof. Let T be a supersubbase for an open subset of K. If rm is covered by a finite number $U_1, \ldots, U_k$ of members of the subbase S and one of these members is equal to $rm \setminus \{y\}$, then another member contains y. Hence both these sets cover rm. If all sets $U_1, \ldots, U_k$ are of the form $U_i = V_i \cup (V_i \cap M \times m)$, then there exist V_i and V_p such that $K = V_i \cup V_p$, and hence $rm = U_i \cup U_p$.

THEOREM 1. *Let m be an infinite discrete space and let K be a regular supercompact space with a dense subset of cardinality less than or equal to that of m. Then the compactification rm of m constructed in Lemma* 1 *is a regular supercompact space and is a superextension of m for a suitable subbase T' on m.*

Proof. Let T be a binary closed subbase for K such that the ring generated by T consists of regular closed sets. For $V \in T$ let us take $V' = V \cup (V \cap M \times m)$. The family

$$F = \{V' \colon V \in T\} \cup \big\{\{y\} \colon y \in Y\big\} \cup \big\{rm \setminus \{y\} \colon y \in Y\big\}$$

is a binary subbase for rm, where rm is constructed for K and for some M dense in K, $\operatorname{card} M \leqslant m$, as in Lemma 1. We shall prove that the ring generated by F is regular. First, let us show that

(*) if B is regular closed in K, then B' is regular closed in rm.

Since B is regular closed, we have $B = \operatorname{cl} U$, where U is open in K. Let $V = U \cup (B \cap M \times m)$. Clearly, V is open in rm and $V \subset B'$. Since B' is closed, we have $\operatorname{cl} V \subset B'$ and

$$\operatorname{cl} V = \operatorname{cl} U \cup (B \cap M \times m) = B \cup (B \cap M \times m) = B'.$$

Assume that W is an element of the ring generated by F. Hence $W = W_1 \cup \ldots \cup W_k$, where each W_i is a finite intersection of elements of F, i.e., $W_i = C_1^i \cap \ldots \cap C_{m_i}^i$, where $C_l^i \in F$. If one of these C_l^i is a one-point set, then so is W_i and, therefore, W_i is regular closed.

Let $C_1^i, \ldots, C_p^i \in \{V' \colon V \in T\}$. Then $C_k^i = G_k^i \cup (G_k^i \cap M \times m)$, where $G_k^i \in T$. Hence

$$C_1^i \cap \ldots \cap C_p^i = (G_1^i \cap \ldots \cap G_p^i) \cup (G_1^i \cap \ldots \cap G_p^i \cap M \times m).$$

Since $G_1^i \cap \ldots \cap G_p^i$ is regular closed, it follows from (*) that so is $C_1^i \cap \ldots \cap C_p^i$. Hence

$$W_i = C_1^i \cap \ldots \cap C_p^i \cap rm \setminus \{y_{p+1}\} \cap \ldots \cap rm \setminus \{y_{m_i}\}$$

is regular closed. Thus W, as a sum of regular closed sets, is regular closed.

Since m is homeomorphic to Y and rm is regular supercompact, it follows from the fact that the regular supercompact space is a superextension of each of its dense subsets that $rm = \lambda_{T'} m$ for some subbase T' of m.

THEOREM 2. *For each supercompact space K there exists a closed subbase S on a discrete space m, where* $\operatorname{card} K = m$, *such that K is homeomorphic to the space of all free maximal linked subfamilies on m relative to the subbase S.*

Proof. Applying Lemma 1 to $M = K$, we get K as the remainder in a compactification rm of a discrete space m of the same cardinality as K. From Lemma 2 it follows that rm has a binary subbase P. Clearly, this subbase satisfies the assumption of the Theorem of van Mill for rm and m. Therefore, rm is a superextension of m for $S = P|m$ and $K = rm \setminus m$. From Lemma 1 it follows that each fixed maximal linked subfamily from S contains a one-point closed-open set. Hence the space of all fixed maximal linked subfamilies is m and $K = rm \setminus m$ is the space of all free maximal linked subfamilies relative to the subbase S.

LEMMA 3. *If K has a normal subbase, then also rm has a normal subbase.*

Proof. Let T be a normal subbase on K. Take

$$S = \{V \cup (V \cap M \times m) \colon V \in T\} \cup \{rm \setminus \{y\} \colon y \in Y\} \cup \{\{y\} \colon y \in Y\}.$$

We shall show that S is a normal subbase. For this purpose take $U, V \in S$ with $U \cap V = \emptyset$ and $U \cup V \neq rm$.

Consider two cases:

(a) Either U or V is a one-point set.

Let $U = \{y\}$. Then $\{y\}$ and $rm \setminus \{y\}$ is the needed pair.

(b) Neither U nor V is a one-point set.

Then there exist $U', V' \in T$ such that

$$U = U' \cup (U' \cap M \times m) \quad \text{and} \quad V = V' \cup (V' \cap M \times m).$$

Since $U \cap V = \emptyset$, we have $U' \cap V' = \emptyset$. From the fact that T is a normal subbase it follows that there exist U_1, V_1 from T such that $U' \subset U_1$, $V' \subset V_1$, $U' \cap V_1 = \emptyset = V' \cap U_1$, and $U_1 \cup V_1 = K$. Hence

$$U \subset \big(U_1 \cup (U_1 \cap M \times m)\big) = H, \qquad V \subset \big(V_1 \cup (V_1 \cap M \times m)\big) = W.$$

The sets H and W form the needed pair.

LEMMA 4. *If K has a T_1-subbase, then rm has a T_1-subbase.*

Proof. Let T be a T_1-subbase on K. We shall prove that S is a T_1-subbase on rm. For this purpose let us take $x \in rm$ and $W \in S$ such that $x \notin W$.

Consider two cases:

(a) If $x \in Y$, then $\{x\} \in S$.

Hence $x \in \{x\}$ and $\{x\} \cap W = \emptyset$.

(b) If $x \notin Y$, then $x \in K$.

If W is a one-point set, then it is sufficient to take $rm \setminus W \in S$. If W is not a one-point set, then $W = V \cup (V \cap M \times m)$, where $V \in T$. Since $x \notin W$, we have $x \notin V$. And since T is a T_1-subbase, there exists a $U \in T$ such that $x \in U$ and $U \cap V = \emptyset$. Hence $U' = U \cup (U \cap M \times m)$ is the needed element of S.

LEMMA 5. *Let K be a T_1-space and S a binary subbase in K. If $F \subset K$ is such that for each $x \notin F$ there exist V and W in S such that $F \subset V \cup W$ and $x \notin V \cup W$, then $S|F = \{T \cap F: T \in S\}$ is a binary subbase in F.*

Proof. Let $R \subset S|F$ be a linked subfamily. Let $\{x_\alpha: \alpha < \beta\}$ be the well ordering of points of $K \setminus F$. Suppose that for all $\alpha < \gamma$, where $\gamma < \beta$, we have defined $S_\alpha \in S$ in such a way that $x_\alpha \notin S_\alpha$ and $R \cup \{S_\alpha \cap F: \alpha < \gamma\}$ is a linked subfamily. In order to define S_γ, take $S_1^\gamma, S_2^\gamma \in S$ such that $F \subset S_1^\gamma \cup S_2^\gamma$ and $x_\gamma \notin S_1^\gamma \cup S_2^\gamma$. Then adjoining $S_1^\gamma \cap F$ or $S_2^\gamma \cap F$, say $S_1^\gamma \cap F$, to $R \cup \{S_\alpha \cap F: \alpha < \gamma\}$ we get again a linked subfamily (if $S_1^\gamma \cap F \cap R' = \emptyset = S_2^\gamma \cap F \cap R''$ for some $R', R'' \in R \cup \{S_\alpha \cap F: \alpha < \gamma\}$, then, since $F \subset S_1^\gamma \cup S_2^\gamma$, we have $R' \cap R'' = \emptyset$ and, therefore, $R \cup \{S_\alpha \cap F: \alpha < \gamma\}$ could not be a linked subfamily) and put $S_\gamma = S_1^\gamma$; the induction step is completed.

Let $R \subset S$ be such that $R = \{F \cap S: S \in R\}$. Then $R \cup \{S_\alpha: \alpha < \beta\}$ is a linked subfamily in S and, therefore,

$$\bigcap R \cap \bigcap \{S_\alpha: \alpha < \beta\} \neq \emptyset.$$

Since $\bigcap \{S_\alpha: \alpha < \beta\} \subset F$, we have

$$\emptyset \neq \bigcap R \cap \bigcap \{S_\alpha: \alpha < \beta\} \subset F.$$

This shows that each linked subfamily in $S|F$ has a non-empty intersection, so $S|F$ is a binary subbase.

From Lemma 5 it follows that $\lambda m \setminus m$, where m is a discrete space, is supercompact (m being the subspace of λm consisting of all fixed maximal linked subfamilies). We shall show that this space $\lambda m \setminus m$ is universal for some class of supercompact spaces in the following sense:

THEOREM 3. *If a supercompact space K has a binary normal T_1-subbase T, then K is a continuous image of $\lambda m \setminus m$, where m is a discrete space and* $\operatorname{card} K = m$.

Proof. Applying Lemma 1 to $M = K$, we get K as the remainder in a compactification rm of a discrete space m of the same cardinality as K. Since K has a normal T_1-subbase, it follows from Lemmas 3 and 4 that rm has a subbase S with the same properties. It is then clear that $Y|S = S'$ is a normal T_1-subbase. It follows from the Theorem of van Mill that rm is a superextension of Y relative to the subbase S.

Let us take a homeomorphism f from m onto Y. By the Theorem of Jensen, there exists an extension $\bar{f}\colon \lambda m \to rm$, where $\bar{f}$ is defined by $\bar{f}(H) = G$, G being a maximal linked subfamily such that G contains a subset $G(H) = \{V \in S\colon f^{-1}(V) \in H\}$. It is clear that $G(H)$ is contained in precisely one maximal linked subfamily.

We show that if H is a free maximal linked subfamily in m, then $\bar{f}(H)$ is free in rm. Let us take $H \in \lambda m$ such that $\bar{f}(H) \in Y$. Assuming that H is free, we have $f^{-1}(y) \in M$ for each $y \in Y$. Since $Y\setminus\{y\} \in S$ for every $y \in Y$ and, by assumption that H is a free maximal linked subfamily, $f^{-1}(Y\setminus\{y\}) \in H$ for each y, we infer that $Y\setminus\{y\} \in G(H)$ for every $y \in Y$. Thus $G(H)$ is free, and hence $G \neq \bar{f}(H)$. A contradiction. Therefore $\bar{f}(\lambda m\setminus m) = K$.

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