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A NOTE ON CONVERGENCE TO INFINITY OF FOURIER SERIES

BY

A. OLEVSKIĬ (MOSCOW)

It is well known that in many problems concerning trigonometric series some compact sets of Lebesgue measure zero exhibit the same behaviour as sets of positive measure. For example in uniqueness theory: there exist compact sets of measure zero which can support Borel measures μ such that $\widehat{\mu}(n) \to 0$ as $|n| \to \infty$ (see A. Zygmund [4], Ch. IX).

Recently S. Konyagin [1] proved that if $E \subset \mathbf{T}$ has positive measure then no trigonometric series converges to $+\infty$ at every point of E. He conjectured that this result can be extended to some compact sets of measure zero. The purpose of this note is to show that this is not so even for L^2-Fourier series.

THEOREM. *Let E be a compact set of measure zero in the circle $\mathbf{T}$. Then there exists a function $f : \mathbf{T} \to \mathbf{R} \cup \{+\infty\}$ satisfying the following conditions*:

(i) *f is $+\infty$ on E and continuous outside E*;

(ii) *$f \in L^p(\mathbf{T})$ for every $p < +\infty$*;

(iii) *the Fourier series $f \sim \sum_{n \geq 0} a_n \cos nt + b_n \sin nt$ converges to f everywhere.*

The proof is based on Carleson's famous theorem. Let U_A denote the set of continuous functions f on $\mathbf{T}$ with Fourier series

$$f(t) = \sum_{n \geq 0} \widehat{f}(n) e^{int}$$

uniformly convergent to f. It is a Banach space with the norm

$$\|f\| = \sup_N \Big\| \sum_{n=0}^{N} \widehat{f}(n) e^{int} \Big\|_{C(\mathbf{T})}.$$

S. Vinogradov showed that the Carleson–Hunt inequality implies some nontrivial lower estimate of the norm of a linear functional of Cauchy integral type on U_A. Using this D. Oberlin [2] proved that for any compact set $E \subset \mathbf{T}$ of measure zero the restriction operator $I : U_A \to C(E)$ is surjective. Hence by Banach space theory every $f \in C(E)$ can be extended to $f^* \in U_A$ so that

$$\|f^*\|_{U_A} \le K\|f\|_{C(E)}$$

where K is a constant depending only on E. We refer to [3] for more details.

Now we give the proof of the theorem given above.

1. For a given natural number ν let us define a trigonometric polynomial f_ν with the following properties:

1.1. $\operatorname{Re} f|_E > 1$;
1.2. $\operatorname{Spec} f \equiv \operatorname{supp} \hat{f} \subset [\nu, \nu'[, \quad \nu' > \nu$;
1.3. $\|f\|_{U_A} < 2K$.

For this purpose by the Oberlin theorem we extend the function $\frac{3}{2}e^{-int}|_E$, then multiply the result by e^{int} and take the partial Fourier sum with a sufficiently large index.

2. For given ν take the polynomials $f_{\nu_1}, f_{\nu_2}, \dots, f_{\nu_s}$ with $\nu_1 = \nu$, $\nu_2 = \nu_1'$, $\nu_3 = \nu_2'$, $\dots$, $\nu_s = \nu_{s-1}'$. Set

$$F_{\nu,s} = \frac{1}{s}\sum_{j=1}^{s} f_{\nu_j}\,.$$

The following assertions are true:

2.1. $\operatorname{Re} F_{\nu,s}|_E > 1$;
2.2. $\operatorname{Spec} F_{\nu,s} \subset [\nu, \bar{\nu}[, \quad \bar{\nu} = \nu_s'$;
2.3. $\|F_{\nu,s}\|_{U_A} < 2K$;
2.4. $\|F_{\nu,s}\|_{L^p(\mathbf{T})} < 2Ks^{-1/p}$ $(p \ge 2)$.

Only the last one needs an explanation.

Denoting by m the normalized Lebesgue measure on $\mathbf{T}$ and using the orthogonality of the polynomials f_{ν_j} we have

$$\int_{\mathbf{T}} |F_{\nu,s}|^2\,dm = \frac{1}{s^2}\sum_{j=1}^{s}\int_{\mathbf{T}} |f_{\nu_j}|^2\,dm \le \frac{1}{s}\max\|f_{\nu_j}\|_{U_A}^2 < \frac{(2K)^2}{s}\,.$$

Then

$$\|F_{\nu,s}\|_{L_m^p(\mathbf{T})}^p \le \int_{\mathbf{T}} |F_{\nu,s}(t)|^2\,dm \cdot \|F_{\nu,s}\|_{C(\mathbf{T})}^{p-2} < \frac{(2K)^p}{s}\,.$$

3. It is easy to define a sequence of real trigonometric polynomials $\{\tau_l\}$ satisfying

3.1. $\|\tau_l\|_{A(\mathbf{T})} \equiv \sum_{n\in\mathbf{Z}} |\hat{\tau}_l(n)| = o(l)$;
3.2. $\tau_l|_E > 1$;
3.3. $\tau_l(t) = o_t(1/l)$ $(t \notin E)$ and o is uniform on every closed segment contained in $\mathbf{T}\setminus E$.

To this end divide the circle $\mathbf{T} = [-\pi, \pi]$ into l segments of equal length. The symbol A_l will denote the union of segments having nonempty intersection with E. Let B_l be the union of segments which are more than $2\pi/l$ distant from A_l. Putting $\varphi_l(t) = 1$ for $t \in A_l$ and $\varphi_l(t) = 0$ for $t \in B_l$ and extending by linear interpolation on the remaining segments we get a Lipschitz function φ_l satisfying

$$\|\varphi_l\|_{A(\mathbf{T})} = O(\|\varphi_l'\|_{L^2(\mathbf{T})}) \qquad \text{(see [4], Ch. VI).}$$

The last expression is $o(l)$.

It is clear that $\bigcap_l A_l = E$, $\bigcup_N \bigcap_{l \geq N} B_l = CE$, so by approximating $2\varphi_l$ by partial Fourier sums with large index we obtain polynomials $\{\tau_l\}$ for which 3.1–3.3 hold.

4. Now suppose

$$F(t) = \sum_l \frac{1}{l} F_{\nu_l, 2^l}(t)\tau_l(t) \equiv \sum u_l(t)\,. \tag{1}$$

We choose the numbers ν_l so as to fulfil the following (see 2.2):

$$\operatorname{Spec} u_l \subset [q_l, q_{l+1}[, \quad 0 < q_1 < q_2 < \dots$$

By 2.4 and 3.1 we have

$$\|u_l\|_{L^p(\mathbf{T})} \leq l^{-1} \|F_{\nu_l, 2^l}\|_{L^p(\mathbf{T})} \|\tau_l\|_{A(\mathbf{T})} = o(2^{-l/2})\,,$$

so $F \in L^p \ \forall p < \infty$.

Similarly by 2.3 and 3.1 we get

$$\|u_l\|_{U_A} \leq l^{-1} \|F_{\nu_l, 2^l}\|_{U_A} \|\tau_l\|_{A(\mathbf{T})} = o(1)\,. \tag{2}$$

Further by 2.3 and 3.3 the series (1) is convergent uniformly on every closed segment contained in $\mathbf{T} \setminus E$. Put

$$f(t) = \begin{cases} \operatorname{Re} F(t)\,, & t \notin E, \\ +\infty\,, & t \in E. \end{cases}$$

It is clear that (i), (ii) are fulfilled.

The partial Fourier sums $S_n(f;t)$ satisfy the condition

$$S_{q_n}(f;t) = \sum_{l=1}^{n-1} \operatorname{Re} u_l(t) \to f(t) \quad \forall t \in \mathbf{T}$$

(see 2.1, 3.2).

Finally, for $q \in [q_l, q_{l+1}[$ we have

$$S_q(f;t) = S_{q_l}(f;t) + S_q(u_l;t) = S_{q_l}(f;t) + o(1)$$

by (2). This completes the proof.

Remarks. 1. It is evident that the condition (ii) in the theorem cannot be replaced by $f \in L^\infty$.

2. The author does not know whether a direct construction not using Carleson's theorem is possible.

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MOSCOW INSTITUTE OF ELECTRONIC ENGINEERING (MIEM)
BOL. VUZOVSKI 3/12, 109028 MOSCOW, U.S.S.R.

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