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CENTRAL POINTS OF CONVEX SETS

BY

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Assume that K is an infinite-dimensional compact convex subset of the Hilbert space l_2. The classical theorem of Keller [7] says that K is homeomorphic to the Hilbert cube $Q = [-1;1]^\infty$. Anderson [1] has proved that $P = (-1;1)^\infty$, the pseudointerior of the Hilbert cube, is homeomorphic to l_2. The problem arises to identify those subsets of K which topologically are l_2 and, in particular, to study subsets $A \subset K$ such that

$$(*) \qquad (K, A) \cong (Q, P),$$

i.e. there is a homeomorphism of K onto Q which carries A onto P. In practical situations, sets A with property $(*)$ arise frequently as complements of the aureoles (see formula (1)) of certain points $y \in K$. The purpose of this paper is to study such points y.

1. Preliminaries. The symbols $\cup$, $\cap$ and $\setminus$ denote the set-theoretical operations of union, intersection and difference, respectively; $+$, $\cdot$ and $-$ being reserved for denoting the algebraic operations on scalars and vectors and also on sets of scalars and vectors. All vector spaces appearing in this paper are over the field R of reals.

For y, z in a vector space, we let

$$[y; z) = \{tz + (1-t)y : 0 \leqslant t < 1\}$$

to be the half-open segment; similarly we denote by $[y; z]$ and $(y; z)$ the closed and the open segments, respectively.

By *map* we always mean a continuous mapping.

A. Assume that W is a convex set in a vector space E. By the *aureole of a point* $y \in W$ we mean the set

$$(1) \qquad \operatorname{aur}_y W = \bigcup_{z \in W} [y; z).$$

Notice that in terms of the aureoles one can easily define the *radial interior* ([3], p. 162) of W,

$$\text{(2)} \qquad \operatorname{rint} W = \bigcap_{y \in W} \operatorname{aur}_y W,$$

and the *set of extreme points*

$$\text{(3)} \qquad \operatorname{Ext} W = \bigcap_{y \in W} (W \setminus \operatorname{aur}_y W \cup \{y\}).$$

1.1. PROPOSITION. (i) $z \in \operatorname{aur}_y W$ *implies* $\operatorname{aur}_z W \subset \operatorname{aur}_y W$; (ii) *if* $y \in \operatorname{rint} W$, *then* $\operatorname{aur}_y W = \operatorname{rint} W$.

Proof of (i). Assume that $y \in W$, $z \in \operatorname{aur}_y W$ and $x \in \operatorname{aur}_z W$. Ignoring the trivial case where x, y, z are co-linear, denote by B the interior of the triangle $\operatorname{conv}\{x, y', z'\}$ such that

$$[z; z'] \cup [y; y'] \subset W, \qquad z \in (y; y'), x \in (z; z'),$$

and let $p \in W$ be a point with the property

$$(y; p) = B \cap (y + R \cdot (x - y))$$

(see Fig. 1). Since $x \in (z; z') \cap B$, we have $x \in [y; p) \subset \operatorname{aur}_y W$.

Statement (ii) follows immediately from (i) and from formula (2).

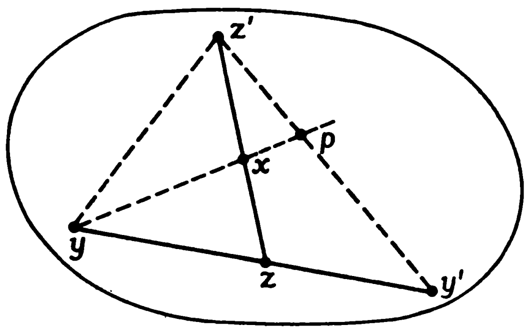

Fig. 1

B. Assume that U is a convex subset of a Banach space X. Denote by X^* the *dual of* X, i.e. the space of all continuous linear functionals on X.

An element $z \in U$ is a *support point* if there is an $f \in X^*$ which supports U at z, i.e.

$$\inf f(U) < \sup f(U) = f(z) = 1.$$

A support point z for U is *strictly exposed* if every $f \in X^*$ which supports U at z is strictly less than 1 on $U \setminus \{z\}$. The sets of all support points and of all strictly exposed points for U will be denoted by $\operatorname{su} U$ and $\operatorname{sex} U$, respectively.

The set U is called *elliptically convex* [7] if $\operatorname{su} U = \operatorname{sex} U$.

C. Assume that V is a closed convex subset of the Hilbert space l_2. Let $s\colon l_2 \to V$ be the *metric projection*, that is the mapping which sends any point $x \in l_2$ to the point y in V which is the nearest to x (¹). For every $u \in l_2$, define the map $g_u\colon V \to V$ by $g_u(x) = s(x+u)$ for $x \in V$.

1.2. PROPOSITION. *Let* $u \in R\cdot(V-V)\setminus(V-V)$. *Then*

(a) $g_u(V) \subset \operatorname{su} V$;

(b) *if* z *and* $z+au$ *are in* V *for some* $a > 0$, *then* $g_u(V) \subset V\setminus \operatorname{aur}_z V$;

(c) *if* $z \in V\setminus \operatorname{su} V$, *then* $g_u(V) \subset V\setminus \operatorname{aur}_z V$;

(d) *if* $z \in \operatorname{sex} V$, *then* $g_u(V) \subset \{z\}\cup(V\setminus \operatorname{aur}_z V)$.

Proof. (a) Let $x_0 \in V$, and $p = g_u(x_0)$. Since p is the nearest point in V for x_0+u, the hyperplane passing through p and orthogonal to x_0+u-p supports V at p.

(b) Let $x \in V$, and $p = g_u(x)$. Pick an $f \in (l_2)^*$ which supports V at p. Clearly, the hyperplane $f^{-1}(1)$ separates the point $x+u$ from the set V, whence $f(x+u) > 1 \geqslant f(x)$. Therefore, $f(u) > 0$ and, remembering that $z+au \in V$, we get

$$f(z) < f(z)+af(u) = f(z+au) \leqslant 1.$$

Hence, for every $c > 0$,

$$f\big(p+c(p-z)\big) = 1+c\big(1-f(z)\big) > 1, \quad \text{i.e.} \quad p+c(p-z) \notin V.$$

Thus $p \in V\setminus \operatorname{aur}_z V$.

(c) follows immediately from the fact that if $z \in V\setminus \operatorname{su} V$, then $\operatorname{aur}_z V \subset V\setminus \operatorname{su} V$.

(d) Suppose that $x \in V$ and

$$p = g_u(x) \notin \{z\}\cup(V\setminus \operatorname{aur}_z V), \quad \text{i.e.} \quad p \in \operatorname{aur}_z V\setminus\{z\}.$$

Then there exists a $y \in V\setminus\{z\}$ such that $p \in (y; z)$. Hence, if $f \in (l_2)^*$ supports V at p, we must have $1 = f(p) = f(y) = f(z)$, so z cannot be strictly exposed in V.

D. Let $Y = (Y, d)$ be a complete metric space, and $Q = [-1;1]^\infty$ the Hilbert cube. A closed subset [an F_σ-subset] A of Y is called a *Z-set* (²) [a Z_σ-*set*] if it satisfies the following condition:

(z) every map $f\colon Q \to Y$ is a uniform limit of $(Y\setminus A)$-valued maps.

(¹) If V is compact, then the standard Bolzano-Weierstrass argument shows that such a y always exists and s is continuous. The same is true for an arbitrary closed convex set, but the proof is more involved (see [6], Chapter V, § 1.4). For an interesting discussion of metric projections in a general metric space setting see Singer [10].

(²) The concept of a Z-set has been introduced by Anderson [2]. The present definition is a modification (due to Toruńczyk [11]) of the Anderson original definition. The two definitions are equivalent in the case of ANR spaces.

The classes of all Z-sets and Z_σ-sets will be denoted by $Z(Y)$ and $Z_\sigma(Y)$, respectively.

1.3. Remark. Let Y be homeomorphic to Q. Then a closed set $A \subset Y$ is a Z-set if and only if, for every positive integer m, there is a map $f\colon Y \to Y \setminus A$ such that

$$\sup_{x \in Y} d\big(f(x), x\big) < \frac{1}{m}.$$

The Baire category argument applied in the space of maps yields

1.4. PROPOSITION. *$A \in Z_\sigma(Y)$ if and only if A is the countable union of Z-sets.*

2. Central points. In this section we assume that K is an infinite-dimensional compact convex subset of l_2.

Definition. A point $y \in K$ is said to be *central* if $\mathrm{aur}_y K \in Z_\sigma(K)$. The set of all central points of K will be denoted by $\mathrm{cent}\,K$.

Central points have been introduced in [4], and their significance is due to the following fact ([4], 2.4, cf. [5], Chapter V, Proposition 4.2):

GENERALIZED KELLER THEOREM. *If $x \in \mathrm{cent}\,K$, then there exists a homeomorphism of K onto Q which carries $K \setminus \mathrm{aur}_x K$ onto P.*

We recall that $P = (-1; 1)^\infty$, the pseudointerior of the Hilbert cube Q.

2.1. PROPOSITION. *If $x \in \mathrm{cent}\,K$, then $\mathrm{aur}_x K \subset \mathrm{cent}\,K$.*

Proof. For every $y \in K$, the set $\mathrm{aur}_y K$ is of type F_σ as the union of the sequence of $(1-1/n)$-homothets of K with respect to y. Now, if $y \in \mathrm{aur}_x K$, then, by Proposition 1.1 (i), $\mathrm{aur}_y K \subset \mathrm{aur}_x K$. We complete the proof by observing that every F_σ-subset of a Z_σ-set is a Z_σ-set itself.

For $u \in l_2$ write

$$\mathrm{diam}_u K = \sup\{\|y\|\colon\ y \in K-K,\ u \in R\cdot y\}.$$

Obviously, $\mathrm{diam}_u K > 0$ iff $u \in R\cdot(K-K)$.

2.2. LEMMA. *There are $u_n \in R\cdot(K-K)\setminus(K-K)$ for $n = 1, 2, \ldots$ such that*

$$\lim_n \mathrm{diam}_{u_n} K = 0.$$

Proof. Let (u_n) be an orthonormal basis for the infinite-dimensional pre-Hilbert space $R\cdot(K-K)$. The set $K-K$ is compact, therefore every bounded sequence of its elements contains a convergent subsequence, and every orthogonal sequence in $K-K$ tends to zero. Hence

$$\lim_n \sup\{\|tu_n\|\colon\ tu_n \in K-K,\ t \in R\} = 0, \quad \text{i.e.} \quad \lim_n \mathrm{diam}_{u_n} K = 0.$$

From the last condition and the fact that u_n's are normalized it follows that $u_n \notin K-K$ for all but finitely many n's.

Using the last lemma and the properties of maps g_u stated in 1.C we prove

2.3. PROPOSITION. $\operatorname{cent} K \supset (K \setminus \operatorname{su} K) \cup \operatorname{sex} K$.

Proof. Suppose that $z \in (K \setminus \operatorname{su} K) \cup \operatorname{sex} K$. By Lemma 2.2, there are $u_n \in R\cdot(K-K)\setminus(K-K)$ such that $\operatorname{diam}_{u_n} K < 1/n$ for $n = 1, 2, \ldots$ Let

$$v_n = n^{-1}\|u_n\|^{-1}u_n.$$

Clearly, $v_n \in R\cdot(K-K)\setminus(K-K)$, whence, by 1.2,

$$g_{v_n}(K) \subset (K \setminus \operatorname{aur}_z K) \cup \{z\}. \tag{4}$$

Evidently, $\|g_{v_n}(x)-x\| \leqslant 2\|v_n\| = 2/n$ for $x \in K$, $n = 1, 2, \ldots$ Hence, for every map $f\colon Q \to K$, the sequence $(g_{v_n} \circ f)$ converges uniformly to f. Thus, by (4), the set $\operatorname{aur}_z K \setminus \{z\}$ satisfies condition (z) and, being of type F_σ, it is a member of $Z_\sigma(K)$. By 1.4, $\operatorname{aur}_z K \in Z_\sigma(K)$, i.e. z is a central point.

2.4. COROLLARY. $\operatorname{rint} K \subset \operatorname{cent} K$.

This follows from Proposition 2.3 and the fact that $\operatorname{rint} K \subset K \setminus \operatorname{su} K$.

2.5. PROPOSITION ([4], 2.6). *Suppose that a point $z \in K$ satisfies one of the following conditions:*

(a) $\inf\{\operatorname{diam}_u K\colon u \in K-K, z+u \in K\} = 0$;

(b) *there is an orthogonal sequence (v_n) such that $v_n \neq 0$, $z+v_n \in K$ for $n = 1, 2, \ldots$*

Then $z \in \operatorname{cent} K$.

Proof. The sufficiency of (a) follows from Proposition 1.2 (b). The implication (b) $\Rightarrow$ (a) is a consequence of the fact that the sequence $(\operatorname{diam}_{v_n} K)$ tends to zero (cf. the proof of Lemma 2.2).

If $\{z_n\colon n = 1, 2, \ldots\}$ is a dense subset of the set K, then, evidently, $z = \sum_{n=1}^{\infty} 2^{-n} z_n$ is not a support point ([3]) for K. Hence, by 2.4, $\operatorname{cent} K$ is non-empty. In fact, a much stronger result holds:

2.6. THEOREM. *The set $\operatorname{cent} K$ is a dense G_δ in K; moreover,*

$$K \setminus \operatorname{cent} K \in Z_\sigma(K).$$

This theorem is a straightforward consequence of the next two lemmas.

2.7. LEMMA. *The set $K \setminus \operatorname{cent} K$ satisfies condition* (z) *as a subset of the space K.*

([3]) This argument is due to E. Michael, see Klee [8]. The fact that $K \setminus \operatorname{su} K$ is non-empty was known to Keller [7].

Proof. Assume that

$$x_0 \in \operatorname{cent} K. \tag{5}$$

Clearly, for each map $f\colon Q \to K$, the maps $f_n\colon Q \to K$ defined by

$$f_n(x) = x_0 + \left(1 - \frac{1}{n}\right)\left(f(x) - x_0\right), \qquad x \in Q,$$

converge uniformly to f and have their values in $\operatorname{aur}_{x_0} K$.

By (5) and 2.1, $\operatorname{aur}_{x_0} K \subset \operatorname{cent} K$. Hence $K \setminus \operatorname{cent} K$ satisfies condition (z).

2.8. LEMMA. *The set* $\operatorname{cent} K$ *is of type* G_δ.

Proof. For each $x \in K$, let

$$K_n(x) = \frac{x}{n} + \left(1 - \frac{1}{n}\right)\cdot K$$

be the $(1-1/n)$-homothet of K with respect to x. For each pair (n, m) of positive integers, denote by L_{nm} the set of points $x \in K$ for which there exists a map $f\colon K \to K$ with the properties

$$\sup_{z \in K} \|f(z) - z\| < \frac{1}{m} \quad \text{and} \quad f(K) \subset K \setminus K_n(x).$$

By the theorem of Keller [7], $K \cong Q$. Hence, by Remark 1.3,

$$\bigcap_m L_{nm} = \{x \in K\colon K_n(x) \in Z(K)\}.$$

Observe that $\bigcap_n K_n(x) = \operatorname{aur}_x K$. Hence, by Proposition 1.2,

$$\begin{aligned}\bigcap_n \bigcap_m L_{nm} &= \{x\colon K_n(x) \in Z(K) \text{ for } n = 1, 2, \ldots\} = \{x\colon \operatorname{aur}_x K \in Z_\sigma(K)\} \\ &= \operatorname{cent} K.\end{aligned}$$

To complete the proof we notice that each L_{nm} is open in K.

In general, $\operatorname{cent} K$ need not coincide with K (see Examples 3.6 and 3.7). However, it follows from Proposition 2.3 that

2.9. COROLLARY. *If K is elliptically convex, then* $\operatorname{cent} K = K$.

3. Comments and examples. Suppose that U is a convex set in a Hausdorff topological vector space X. Let $A(U)$ denote the set of all continuous affine functionals $f\colon A \to R$. Finally, let $\operatorname{asu} U$ and $\operatorname{asex} U$ be the sets of the points in U which are supported and which are strictly exposed, respectively, by functionals in $A(U)$.

Since every $f \in X^*$ restricts to a member of $A(U)$, we have

$$\operatorname{su} U \subset \operatorname{asu} U$$

and

$$\operatorname{sex} U \supset \operatorname{su} U \cap \operatorname{asex} U \quad \text{or} \quad \operatorname{asex} U \subset \operatorname{sex} U \cup (U \setminus \operatorname{su} U).$$

Hence

3.1. PROPOSITION. $(U\setminus \operatorname{su} U) \cup \operatorname{sex} U \supset (U \setminus \operatorname{asu} U) \cup \operatorname{asex} U$.

We also have

3.2. PROPOSITION. $U \setminus \operatorname{rint} U \supset \operatorname{asu} U \supset \operatorname{su} U$.

In fact, if $f \in A(U)$ supports U at x, and $y \in U$ is such that $f(y) < f(x)$, then there is no $z \in U$ with $x \in (y; z)$ (because otherwise we would have $f(z) > f(x) = \sup f(U)$), i.e. $x \in U \setminus \operatorname{rint} U$.

Wojtaszczyk [12] has shown that if U is a closed convex subset of a Banach space and $U - U$ spans linearly the whole space, then $U \setminus \operatorname{rint} U = \operatorname{su} U$ if and only if U has non-empty topological interior.

By 2.4 and by the Generalized Keller Theorem we have

3.3. COROLLARY. $(K, K\setminus \operatorname{rint} K) \cong (Q, P)$ *provided that* $\operatorname{rint} K$ *is non-empty.*

We note that there exist compact convex sets K with $\operatorname{rint} K = \emptyset$.

3.4. Example ([4], 2.13). *Let M denote the set of all probabilistic measures on $[0; 1]$. Then $\operatorname{rint} M = \emptyset$. The M regarded as a subset of $C([0;1])^*$ equipped with the weak-star topology is compact and admits an affine-homeomorphic embedding into l_2.*

Proof. Take an arbitrary $\mu \in M$. Evidently, there is a $t \in [0; 1]$ with $\mu(\{t\}) = 0$. Let δ_t be the Dirac measure at t, i.e. $\delta_t(A) = 1$ if $t \in A$ and $\delta_t(A) = 0$ if $t \notin A$. If $\nu \in C([0;1])^*$ is such that $\mu \in (\delta_t; \nu)$ and, say, $\mu = (1-a)\delta_t + a\nu$ $(0 < a < 1)$, then

$$\nu(\{t\}) = \frac{a-1}{a} < 0, \quad \text{i.e.} \quad \nu \notin M.$$

Hence $\mu \notin \operatorname{rint} M$. Thus $\operatorname{rint} M = \emptyset$.

The compactness of M follows from the fact that it is closed (with respect to the weak-star topology) in the unit ball of $C([0;1])^*$.

The required embedding $h\colon M \to l_2$ can be defined by $h(\mu) = (\mu(x_n))$, where (x_n) is a fixed sequence in $C([0;1])$ whose linear span is dense in the space and such that

$$\sum_n \|x_n\|^2 < \infty$$

(for instance, $x_n(t) = t^{n-1}/n$ for $t \in [0; 1]$).

An arbitrary infinite-dimensional compact convex set (in a topological vector space) which admits an affine-homeomorphic embedding into l_2 is called ([4], § 2) a *Keller space*. Let us remark ([8]; cf. [5], Chapter III, § 2) that every infinite-dimensional compact convex set which admits a countable separating family of continuous affine functionals is a Keller space. (The embedding into l_2 can be constructed as in the proof of Example 3.4.)

3.5. Remark. Statements 2.1, 2.3, 2.4, 2.5 (a) and 2.6 are valid for arbitrary Keller spaces if one replaces $\operatorname{su} K$ and $\operatorname{sex} K$ by $\operatorname{asu} K$ and $\operatorname{asex} K$, respectively; the $\operatorname{diam}_u K$ can be understood as $\sup\{d(x, y)\colon x, y \in K,\ u \in R\cdot(x-y)\}$, where d is an arbitrary metric compatible with the topology of the Keller space K.

This follows from Proposition 3.1 and from the affine-topological invariance of the notions: cent, asu, asex, rint.

From the Definition given in Section 2 it follows that if $x \in \operatorname{cent} K$, then

(6) $K \setminus \operatorname{aur}_x K$ is dense in K.

This fact will be used in the next three examples.

3.6. Example. *Let*

$$A = \left\{x \in l_2\colon x_1 = 1,\ |x_i| \leqslant \frac{1}{i} \text{ for } i \geqslant 2\right\}.$$

If $K = \operatorname{conv}(A \cup \{0\})$, *then* $\operatorname{cent} K = K \setminus \{0\}$.

In fact, the points of $K \setminus \{0\}$ are central by Proposition 2.5 (b). The set $K \setminus \operatorname{aur}_0 K = A$ does not satisfy (6).

3.7. Example. *Let A be the same as in Example* 3.6, *and let*

$$B = \left\{x \in l_2\colon x_1 = 0,\ |x_i| \leqslant \frac{1}{m} \text{ for } 2 \leqslant i \leqslant m \text{ and } x_i = 0 \text{ for } i > m\right\},$$

where m is a fixed integer not less than 2. *Then, for* $K = \operatorname{conv}(A \cup B)$, *we have* $K \setminus \operatorname{cent} K = B$.

For the proof observe that the points in $K \setminus B$ satisfy condition (b) of 2.5, and if $x \in B$, then the closure of $K \setminus \operatorname{aur}_x K$ does not contain the point 0.

3.8. Example (H. Toruńczyk). *Let*

$$K = \left\{x \in l_2\colon x_i \geqslant 0 \text{ for } i = 1, 2, \ldots, \sum_{i=1}^{\infty} x_i \leqslant 1\right\}.$$

Then the set $K \setminus \operatorname{cent} K$ *is infinite-dimensional.*

Proof. Assume that

$$x \in A_n = \left\{x \in K\colon \sum_{i=1}^{n} x_i = 1\right\} \quad \text{and} \quad y \in K.$$

Then

$$y \in K \setminus \mathrm{aur}_x K \quad \text{iff} \quad \underset{t>0}{\forall}\, y + t(y-x) \notin K \quad \text{iff} \quad \underset{t>0}{\forall}\ \underset{i\leqslant n}{\exists}\, y_i + t(y_i - x_i) < 0$$
$$\text{iff} \quad \underset{i\leqslant n}{\exists}\ \underset{t>0}{\forall}\, (1+t)y_i - tx_i < 0 \quad \text{iff} \quad \underset{i}{\exists}\, y_i = 0,\ x_i \neq 0.$$

Hence, for $x \in A_n$, we have

$$K \setminus \mathrm{aur}_x K = \{y \in K\colon\ \underset{i}{\exists}\, y_i = 0 \neq x_i\} \subset \bigcup_{i=1}^{n} \{y \in K\colon\ y_i = 0\},$$

whence $K \setminus \mathrm{aur}_x K$ is not dense in K and, therefore, $x \notin \mathrm{cent} K$. Thus $K \setminus \mathrm{cent} K$ contains the infinite-dimensional set $\bigcup_{n=1}^{\infty} A_n$ and, therefore, it is infinite-dimensional itself.

The affine homeomorphism $(x_n) \to (x_n/n)$ carries K onto a compact subset of l_2.

3.9. Question. Is condition (6) always equivalent to $x \in \mathrm{cent} K$? **(P 1000)**

3.10. Example. The solid ellipsoid

$$\mathcal{E} = \Big\{x \in l_2\colon\ \sum_n n^2 |x_n|^2 \leqslant 1\Big\}$$

is compact and elliptically convex. Therefore, by Corollary 2.9, $\mathrm{cent}\, \mathcal{E} = \mathcal{E}$. Let $\mathcal{E} \setminus \mathrm{aur}_0 \mathcal{E} = S$ be the "surface" of the ellipsoid. If $y \in S$, then $\mathcal{E} \setminus \mathrm{aur}_y \mathcal{E} = S \setminus \{y\}$.

3.11. Example. Let $\mathcal{E}$ and S be those of Example 3.10. Let

$$\mathcal{E}^+ = \{x \in \mathcal{E}\colon\ x_1 \geqslant 0\} \quad \text{and} \quad S^+ = S \cap \mathcal{E}^+.$$

By Propositions 2.3 and 2.5 (b), we have $\mathrm{cent}\, \mathcal{E}^+ = \mathcal{E}^+$. Evidently, $\mathcal{E}^+ \setminus \mathrm{aur}_0 \mathcal{E}^+ = S^+$.

3.12. Remark. Evidently, the ellipsoid S is homeomorphic to the unit sphere Σ of l_2; $S \setminus \{y\}$ is homeomorphic to Σ with one point removed and, *via* the stereographic projection, to the whole space l_2; S^+ is homeomorphic to the upper half-sphere and to the closed unit ball B of l_2. Hence, using the Generalized Keller Theorem, we obtain the well-known results of Klee [9] and Anderson [1]:

$$l_2 \cong \Sigma \cong B \cong P \cong R^\infty.$$

3.13. Example. *The set of central points of the Hilbert cube Q is the whole Q.*

To show this observe that Q is affinely homeomorphic to the subset $\{x \in l_2\colon\ |x_i| \leqslant 1/i \text{ for } i = 1, 2, \ldots\}$ of l_2 and apply Proposition 2.5 (b).

Examples 3.12 and 3.13 and Corollary 2.9 show that the class of K's for which $\operatorname{cent} K = K$ is wider than that of infinite-dimensional compact elliptically convex subsets of l_2. But even the latter class is quite extensive, as the next proposition shows.

3.14. PROPOSITION ([7]; cf. [5], Chapter III, § 1). *If K is a convex subset of l_2 such that $0 \in K \setminus \operatorname{su} K$, then, for every $\varepsilon > 0$, the map $f: l_2 \to l_2$ defined by $f(x) = x \cdot (1 + \varepsilon \|x\|)^{-1}$ carries K onto an elliptically convex set. Hence the collection of elliptically convex compact subsets of l_2 is dense, in the Hausdorff metric, in the space of all compact convex subsets of l_2.*

By Theorem 2.6, for every K (which is an infinite-dimensional compact convex subset of l_2), almost all points, in the sense of the Baire category, are central. The last proposition substantiates the conjecture that "almost all" sets K have the property $\operatorname{cent} K = K$.

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