COLLOQUIUM MATHEMATICUM

VOL. XL 1979 FASC. 2

ON THE PERMEABILITY OF SUBMEASURES ON FINITE ALGEBRAS

BY

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Let $\mathscr{A}$ be an algebra of subsets of a set X. A set function $\psi: \mathscr{A} \to [0, \infty[$ is called

normalized if $\psi X = 1$,

a *submeasure* if ψ is monotone and subadditive and $\psi\emptyset = 0$,

a *measure* if ψ is finitely additive.

For a submeasure φ, the *permeability* of φ is defined by

$$a(\varphi) = \sup\{\mu X \mid \mu \text{ measure on } \mathscr{A}, \mu \leqslant \varphi\}.$$

Popov ([4], Theorem 2) and Topsøe ([5], (1)) remarked that $a(\varphi)$ can be expressed in terms of multiple coverings of X (see also [3], Satz 2.2). A finite sequence $\mathscr{C} = (A_1, A_2, \ldots, A_m)$ of elements of $\mathscr{A}$ is called a *k-fold exact covering of X* if the characteristic functions of A_i satisfy

$$\sum_{i=1}^{m} 1_{A_i} = k \cdot 1_X.$$

For a k-fold covering $\mathscr{C} = (A_1, A_2, \ldots, A_m)$ let

$$s(\mathscr{C}, \varphi) = \frac{1}{k} \sum_{i=1}^{m} \varphi A_i.$$

THEOREM 1. *There is a measure $\mu' \leqslant \varphi$ with*

$$\mu' X = a(\varphi) = \inf\{s(\mathscr{C}, \varphi) \mid \mathscr{C} \text{ is a multiple exact covering of } X\}.$$

A proof is given in [4].

The simplest example illustrating the theorem is the following (see [4] and [5])

Example 1. Let $X_n = \{1, 2, \ldots, n\}$, $n \geqslant 2$. We define a normalized submeasure φ on $\mathscr{P}(X_n)$ by $\varphi\emptyset = 0$, $\varphi X = 1$, $\varphi A = \frac{1}{2}$ otherwise. Let

$A_i = X_n - \{i\}$ $(i = 1, 2, \ldots, n)$. Then $\mathscr{C}_n = (A_1, A_2, \ldots, A_n)$ is an $(n-1)$-fold covering of X_n and

$$s(\mathscr{C}_n, \varphi) = \frac{n}{2(n-1)}.$$

On the other hand,

$$\mu' A = \frac{\operatorname{card} A}{2(n-1)}$$

defines a measure $\mu' \leqslant \varphi$ with $\mu' X_n = a(\varphi) = s(\mathscr{C}_n, \varphi)$.

Products of the configurations $(X_n, \mathscr{C}_n)$ can be used to show that for every $\varepsilon > 0$ there are a finite set X and a normalized submeasure φ on $\mathscr{P}(X)$ with $a(\varphi) < \varepsilon$, and that there exist non-trivial submeasures on infinite algebras with $a(\varphi) = 0$, so-called pathological submeasures. This was done by Popov ([4], Section 3) — his example seems to be the first and the prettiest one — and by Herer and Christensen [2]. Other constructions are given by Preiss, Vilímovský and Topsøe in [5] and below in Example 3.

On finite algebras, however, pathological submeasures do not exist: if φ is a normalized submeasure on $\mathscr{P}(X_n)$, then $\varphi\{i\} \geqslant 1/n$ for some i, so that $\mu A = 1/n$ for $i \in A$ and $\mu A = 0$ otherwise define a measure $\mu \leqslant \varphi$ with $\mu X_n = 1/n$.

The goal of this paper is to determine (as far as possible) the *minimum permeability of a normalized submeasure on an n-point set*:

$$a_n = \inf\{a(\varphi) \mid \varphi \text{ normalized submeasure on } \mathscr{P}(X_n)\}.$$

It is clear that $a_1 = 1$ and $a_{n+1} \leqslant a_n$ for all n. The remarks above imply

$$\frac{1}{n} \leqslant a_n \leqslant \frac{n}{2(n-1)} \quad (n \geqslant 2) \quad \text{and} \quad \lim a_n = 0.$$

Topsøe [5] gave these relations and asked for more information on the a_n's. L. Vasak also dealt with these numbers. At the 1975 Winter School in Štefanová, Czechoslovakia, he posed the interesting question:

What is the smallest number q with

$$a_q < \frac{q}{2(q-1)}?$$

We think that this number is 11 but we were able only to prove $6 \leqslant q \leqslant 11$.

Example 2.

1	2	3	4	5	6	7				
1	2	3	4				8	9	10	
1	2			5	6		8	9		11
		3	4	5	6		8		10	11
1		3		5		7	8	9	10	
1			4		6	7	8	9		11
	2	3			6	7		9	10	11
	2		4	5		7			10	11

The sets given as lines in this matrix form a 5-fold covering of X_{11}. Any two of them do not cover X_{11}. Assigning to every set the value $\frac{1}{3}$, we generate a normalized submeasure φ on X_{11} with

$$a_{11} \leqslant a(\varphi) \leqslant s(\mathscr{C}, \varphi) = \frac{8}{15} < \frac{11}{20}.$$

For small n, we can determine a_n by considering all multiple exact coverings $\mathscr{C}$ of X_n with the following properties:

(a) $\mathscr{C}$ has no exact subcoverings. (If $\mathscr{C}$ splits into two disjoint exact coverings $\mathscr{C}_1$ and $\mathscr{C}_2$, then $s(\mathscr{C}_i, \varphi) \leqslant s(\mathscr{C}, \varphi)$ for $i = 1$ or $i = 2$.)

(b) $\mathscr{C}$ does not contain disjoint sets. (Replacing disjoint sets A and B of $\mathscr{C}$ by $A \cup B$ we do not enlarge $s(\mathscr{C}, \varphi)$.)

(c) The sets of $\mathscr{C}$ separate the points of X_n. (Otherwise, $\mathscr{C}$ may be realized as a covering of X_m, where $m < n$.)

For every such $\mathscr{C}$ we determine

$$a(\mathscr{C}) = \inf\{s(\mathscr{C}, \varphi) \mid \varphi \text{ normalized submeasure on } \mathscr{P}(X_n)\}$$

and get

$$a_n = \min\{a_{n-1}\} \cup \{a(\mathscr{C}) \mid \mathscr{C} \text{ covering of } X_n \text{ with (a), (b), (c)}\}.$$

X_2 does not have a covering of the desired kind, hence $a_2 = a_1 = 1$. The only covering of X_3 is $\mathscr{C}_3$ (see Example 1). On X_4 there are $\mathscr{C}_4$ and

$$\mathscr{C}_4' = (\{1, 2, 3\}, \{1, 2, 3\}, \{1, 4\}, \{2, 4\}, \{3, 4\}).$$

Consider $\mathscr{C}_n$ and an arbitrary normalized submeasure φ on X_n. Then

$$(n-1)\sum_{i=1}^{n} \varphi A_i = \sum_{i<j} (\varphi A_i + \varphi A_j) \geqslant \sum_{i<j} 1 = \frac{n(n-1)}{2},$$

which implies

$$\sum_{i=1}^{n} \varphi A_i \geqslant \frac{n}{2} \quad \text{and} \quad s(\mathscr{C}_n, \varphi) \geqslant \frac{n}{2(n-1)}.$$

Example 1 verifies that

$$a(\mathscr{C}_n) = \frac{n}{2(n-1)}.$$

The same method applies to $\mathscr{C}_4'$. Three copies of $\mathscr{C}_4'$ can be splitted into 7 families which cover X_4. Thus

$$s(\mathscr{C}_4', \varphi) \geqslant \frac{7}{3\cdot 3}$$

for every normed submeasure φ, the equality holding if

$$\varphi\{1, 2, 3\} = \frac{2}{3} \quad \text{and} \quad \varphi\{i, 4\} = \frac{1}{3}.$$

Hence $a(\mathscr{C}_4') = 7/9$.

Now we are able to determine

$$a_3 = \frac{3}{4} \quad \text{and} \quad a_4 = \min\left\{\frac{3}{4}, \frac{7}{9}, \frac{4}{6}\right\} = \frac{2}{3}.$$

To get $a_5 = 5/8$ we must consider 9 non-isomorphic coverings, and for $n \geqslant 6$ our method becomes too expansive. However, as well as for many combinatorial functions [1], we can establish a rather simple asymptotical estimation. At first we show, roughly speaking, that submeasures with small permeability admit small values on certain large sets and large values on small sets.

PROPOSITION 1. *Let $\mathscr{A}$ be an algebra of subsets of X, φ a submeasure, and μ a measure on $\mathscr{A}$. Further, let $r, s \in \,]0, \mu X[$ and $\varepsilon > a(\varphi)$.*

(a) *There is a set $B \in \mathscr{A}$ with $\mu B \geqslant r$ and*

$$\varphi B < \varepsilon\left(1+\log\frac{\mu X}{\mu X - r}\right).$$

(b) *There is a set $C \in \mathscr{A}$ with $\mu C \leqslant s$ and*

$$\varphi C > \varphi X - \varepsilon\left(1+\log\frac{\mu X}{s}\right).$$

For illustration, let $X = X_n$, φ normalized, μ the counting measure, and $r = s = n/2$. With $\varepsilon = 1.1\,a(\varphi)$ we get sets B and C satisfying $\operatorname{card} B = \operatorname{card} C = [n/2]$ and $\varphi B < 2a(\varphi)$, $\varphi C > 1-2a(\varphi)$. Thus, for $a(\varphi) \leqslant \frac{1}{4}$, we have $\varphi B \neq \varphi C$. If $a(\varphi)$ is small, then φ becomes very asymmetric in the sense that sets of equal cardinality have φ-values near zero and one (cf. [5], Proposition 1).

Proof. There is a k-fold covering $\mathscr{C} = (A_1, A_2, \ldots, A_m)$ of X with $s(\mathscr{C}, \varphi) < \varepsilon$. If $B \in \mathscr{A}$ and $B \neq X$, there exists an i with

(i) $$\varphi A_i < \varepsilon \frac{\mu(A - B)}{\mu(X - B)}.$$

Indeed, the sum over $i = 1, 2, \ldots, m$ taken on the left-hand and right-hand sides of (i) equals $ks(\mathscr{C}, \varphi)$ and $k\varepsilon$, respectively.

Now let

$$B_1 = A_{i_1}, \quad \text{where } \varphi A_{i_1} < \varepsilon \frac{\mu A_{i_1}}{\mu X}.$$

Suppose that B_j is constructed for $j \leqslant p-1$ and $B_{p-1} \neq X$. Then let $B = B_{p-1}$ and let $B_p = B \cup A_{i_p}$, where $A_i = A_{i_p}$ fulfills (i). For some $p \leqslant m$ we get $B_p = X$.

We consider a fixed p and write $x_0 = 0$, $x_j = \mu B_j$ $(j = 1, 2, \ldots, p-1)$ and $a = \mu X$. Further, let x_p be a number with $x_{p-1} < x_p \leqslant \mu B_p$. By the construction we get

$$\varphi B_p = \sum_{j=1}^{p} \varphi A_{i_j} < \varepsilon \left[\sum_{j=1}^{p} \frac{x_j - x_{j-1}}{a - x_{j-1}} + \frac{\mu B_p - x_p}{a - x_{p-1}} \right],$$

where the right-hand sum is majorized by

$$\int_0^{x_p} \frac{dx}{a - x} = \log \frac{a}{a - x_p}.$$

Consequently,

(ii) $$\varphi B_p < \varepsilon \left[\log \frac{\mu X}{\mu X - x_p} + \frac{\mu B_p - x_p}{\mu X - x_{p-1}} \right].$$

Now we choose p in such a way that $\mu B_{p-1} < r \leqslant \mu B_p$. With $x_p = r$ and $B_p = B$, (ii) yields the inequality of (a). To verify (b), we put $r = \mu X - s$, use (a) and let $C = X - B$.

THEOREM 2.

(a) $$\frac{1}{0.41 + \log n} \leqslant a_n \leqslant \frac{2\log 2}{-\log 2 + \log n} \quad \textit{for } n > 2.$$

(b) $$1 \leqslant \liminf a_n \log n \leqslant \limsup a_n \log n \leqslant 2\log 2.$$

Proof. (b) follows immediately from (a). We are going to prove the left-hand inequality of (a). It is true for $n = 2$ and $n = 3$. Let $n > 3$ and $\varepsilon > a_n$. By the definition of a_n and by Theorem 1, there are a normalized submeasure φ on X_n and a k-fold covering $\mathscr{C} = (A_1, A_2, \ldots, A_m)$ of X_n with $s(\mathscr{C}, \varphi) < \varepsilon$. In the construction of the proof above, let $X = X_n$ and let μ be the counting measure ($\mu A = \operatorname{card} A$). Choose p in such a way that $\operatorname{card} B_{p-1} < n-3$ and $q = \operatorname{card} B_p \geqslant n-3$. For $x_p = n-3$ from (ii) we obtain

$$\varphi B_p < \varepsilon\left(\log\frac{n}{3} + \frac{q-(n-3)}{4}\right).$$

Now recall that $a(\varphi) < \varepsilon$. This means that every subset C of X containing one or two points fulfills $\varphi C < \varepsilon$, and every three-point subset D satisfies $\varphi D < (4/3)\varepsilon$. (If this is not true, then there is a measure μ on $\mathscr{P}(D)$ with $\mu \leqslant \varphi$ and $\mu D = a_3\cdot(4/3)\varepsilon = \varepsilon$. But μ can be extended to a measure on $\mathscr{P}(X)$ majorized by φ. This contradicts $a(\varphi) < \varepsilon$.) Since φ is normalized, we have $\varphi B_p + \varphi(X - B_p) \geqslant 1$. Hence we have

for $q = n-3$,

$$\varepsilon\left(\log\frac{n}{3} + \frac{4}{3}\right) \geqslant 1;$$

for $q = n-2$,

$$\varepsilon\left(\log\frac{n}{3} + \frac{1}{4} + 1\right) \geqslant 1;$$

for $q = n-1$,

$$\varepsilon\left(\log\frac{n}{3} + \frac{2}{4} + 1\right) \geqslant 1;$$

for $q = n$,

$$\varepsilon\left(\log\frac{n}{3} + \frac{3}{4}\right) \geqslant 1.$$

In each case,

$$\varepsilon \geqslant \frac{1}{0.41 + \log n}.$$

This holds for every $\varepsilon > a_n$ and, consequently, for $\varepsilon = a_n$.

Let us prove now the right-hand inequality of Theorem 2 (a). For every $n > 2$ there exists an r with $2^r \leqslant n < 2^{r+1}$. We have $a_n \leqslant a_{(2^r-1)}$ and $\log_2 n < r+1$. Example 3 below shows that $a_{(2^r-1)} < 2/r$. Hence

$$a_n < \frac{2}{r} < \frac{2}{-1+\log_2 n} = \frac{2\log 2}{-\log 2 + \log n}.$$

Example 3. Let $V = \{0,1\}^r$ be the r-dimensional linear space over the field $K_2 = \{0,1\}$ and $X = V - \{0\}$. We consider hyperplanes, that means $(r-1)$-dimensional subspaces of V, and use the following two facts:

(1) Every point of X is contained in the same number of hyperplanes.

(2) The intersection of $r-1$ hyperplanes is a one-dimensional linear subspace of V, hence it contains an element of X.

Let $\mathscr{C}$ denote the family of all complements of hyperplanes. From (1) and (2) we directly obtain

(1′) $\mathscr{C}$ is an exact covering of X.

(2′) The union of $r-1$ sets of $\mathscr{C}$ does not cover X.

Assigning to every member of $\mathscr{C}$ the value $1/r$ and to X the value 1, we generate a normalized submeasure φ on $\mathscr{P}(X)$. Let $m = \operatorname{card}\mathscr{C}$ and let k be the multiplicity of $\mathscr{C}$. Since every set of $\mathscr{C}$ contains 2^{r-1} points of X, we have

$$m\cdot 2^{r-1} = k(2^r-1).$$

Hence

$$a_{(2^r-1)} \leqslant s(\mathscr{C},\varphi) = \frac{1}{k}\frac{k(2^r-1)}{2^{r-1}}\frac{1}{r} < \frac{2}{r}.$$

PROBLEM (**P 1057**). Does there exist, for some r, a covering $\mathscr{C}'$ of X which is better than $\mathscr{C}$ in Example 3 in the sense that $a(\mathscr{C}') < a(\mathscr{C})$?

We conjecture that this is not possible, thus

$$\lim a_n \log n = 2\log 2.$$

Let us state now an interesting corollary to Proposition 1.

For a submeasure φ on $\mathscr{A}$, we put

$$\beta(\varphi) = \inf\{\mu X \mid \mu \text{ measure on } \mathscr{A},\ \mu \geqslant \varphi\}$$

and for $A \in \mathscr{A}$

$$\mu' A = \sup\Big\{\sum_{i=1}^{n}\varphi A_i \Big| n \text{ positive integer, } A_i \subseteq A,\ A_i \in \mathscr{A},\ A_i \text{ pairwise disjoint}\Big\}.$$

The following analogue of Theorem 1 is easy to show:

If $\mu' X = \infty$, then there is no measure majorizing φ, and there is a sequence $A_1, A_2, \ldots$ of disjoint elements of $\mathscr{A}$ with $\sum \varphi A_i = \infty$.

If $\mu' X < \infty$, then μ' is a measure, $\mu' \geqslant \varphi$ and $\mu' X = \beta(\varphi)$.

THEOREM 3. *Let φ be a normalized submeasure on $\mathscr{A}$.*

(a) *If $a(\varphi) \neq 0$, then*

$$\beta(\varphi) \geqslant a(\varphi)\exp\left(\frac{1}{a(\varphi)} - 2\right).$$

(b) *If $a(\varphi) = 0$, then there is no measure $\mu \geqslant \varphi$, and there exists a sequence of pairwise disjoint sets $A_1, A_2, \ldots$ of $\mathscr{A}$ with $\sum \varphi A_i = \infty$.*

Proof. Let $\mu \geqslant \varphi$ and $s = \varepsilon > a(\varphi)$. By Proposition 1, there is a C with

$$s \geqslant \mu C \geqslant \varphi C > 1 - \varepsilon\left(1 + \log\frac{\mu X}{s}\right).$$

Consequently,

$$\varepsilon\left(1+\log\frac{\mu X}{\varepsilon}\right) > 1-\varepsilon \quad \text{and} \quad \mu X > \varepsilon\exp\left(\frac{1}{\varepsilon}-2\right).$$

This is true for every $\mu \geqslant \varphi$ and for every $\varepsilon > a(\varphi)$. Thus (a) is proved, and $a(\varphi) = 0$ implies that μX is greater than any finite number. By the remark on μ', the proof is completed.

Part (a) states that small $a(\varphi)$ implies large $\beta(\varphi)$. The converse does not hold: the submeasure φ defined on X_n by $\varphi A = 1$ for $A \neq \emptyset$ satisfies $\beta(\varphi) = n$ and $a(\varphi) = 1$.

Statement (b) is connected with the interesting question whether there are continuous pathological submeasures. By Theorem 2 of [2] and Theorem 4 of [4], this question is equivalent to a well-known problem of Maharam. However, the condition given in (b) is much weaker than discontinuity (consider $\varphi = \sqrt{\lambda}$, where λ denotes Lebesgue measure on [0, 1]).

Acknowledgement. The author is indebted to Prof. S. Hartman and to Dr. Z. Lipecki for several helpful suggestions and important references.

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Reçu par la Rédaction le 29. 4. 1977