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ALGEBRAIC STRUCTURES WITH PRESCRIBED AUTOMORPHISM GROUPS

BY

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Armbrust and Schmidt [1] prove that every group G of permutations of a set A is the group of all automorphisms of an algebra whose base set is A. The algebra which they construct has a single operation whose rank is equal to the cardinal of A, and they observe that their theorem is not true if the operations are required to be of finite rank. Given a cardinal m, we here consider the following two properties of the group G of permutations of A:

$\alpha_m(G)$. *G is the group of all automorphisms of an algebra $\mathbf{A}$ whose base set is A, and all of whose operations are of rank less than m.*

$\beta_m(G)$. *For every permutation φ of A, if for every subset X of A with fewer than m elements there exists a member of G that agrees with φ on X, hen $\varphi \in G$.*

It is almost obvious that $\alpha_m(G)$ implies $\beta_{m+1}(G)$. Using a construction similar to the one in [1] we prove that, except in the case $m = 2$, $\beta_m(G)$ implies $\alpha_m(G)$. Thus for m infinite the properties $\alpha_m(G)$ and $\beta_m(G)$ are equivalent. The case $m = \aleph_0$ is of course the one of greatest interest. For m finite, examples show that neither of the two implications can be reversed, and the problem of characterizing those groups G for which $\alpha_m(G)$ holds is therefore not completely solved (**P 626**).

THEOREM 1. *For any cardinal m, $\alpha_m(G)$ implies $\beta_{m+1}(G)$.*

Proof. Suppose $\mathbf{A} = \langle A, F_i \rangle_{i \in I}$ is an algebra all of whose operations are of rank less than m, and suppose the group of all automorphisms of $\mathbf{A}$ is G. Consider any permutation φ of A that satisfies the hypothesis of $\beta_{m+1}(G)$. Given $i \in I$, and a sequence $x = \langle x_0, x_1, \ldots, x_\varkappa, \ldots \rangle_{\varkappa < \varrho(i)}$, where $\varrho(i)$ is the rank of F_i, the set $X = \{x_\varkappa | \varkappa < \varrho(i)\} \cup \{F_i(x)\}$ has fewer than $m+1$ elements, and therefore there exists a member σ of G

[1] M. Armbrust and J. Schmidt, *Zum Cayleyschen Darstellungssatz*, Mathematische Annalen 154 (1964), p. 70-73.

that agrees with φ on X. Consequently

$$F_i(\varphi x) = F_i(\sigma x) = \sigma\big(F_i(x)\big) = \varphi\big(F_i(x)\big).$$

Thus φ is an automorphism of $\boldsymbol{A}$; i.e., $\varphi \epsilon G$.

THEOREM 2. *For any cardinal $m \neq 2$, $\beta_m(G)$ implies $\alpha_m(G)$.*

Proof. First suppose $m > 2$. Assuming that $\beta_m(G)$ holds, let K be the set of all one-to-one sequences of elements of A having more than one term but fewer than m terms. For $x \epsilon K$ let $\varrho(x)$ be the type of the sequence x, and define an operation F_x of rank $\varrho(x)$ over A by the condition that, for each $\varrho(x)$-termed sequence y of elements of A,

$$F_x(y) = \begin{cases} y_0 & \text{if} \quad y \epsilon Gx, \\ y_1 & \text{if} \quad y \notin Gx. \end{cases}$$

Here Gx is the set of all images σx of the sequence x under the members σ of G. We complete the proof of the theorem by showing that G is the group of all automorphisms of the algebra

$$\boldsymbol{A} = \langle A, F_x \rangle_{x \epsilon K}.$$

Suppose $\varphi \epsilon G$. Given $x \epsilon K$ and a sequence y of type $\varrho(x)$ of elements of A, observe that $y \epsilon Gx$ iff $\varphi y \epsilon Gx$. Therefore, if $y \epsilon Gx$, then

$$F_x(\varphi y) = \varphi(y_0) = \varphi\big(F_x(y)\big),$$

but if $y \notin G$, then

$$F_x(\varphi y) = \varphi(y_1) = \varphi\big(F_x(y)\big).$$

Consequently φ is an automorphism of $\boldsymbol{A}$.

Suppose $\varphi \notin G$. Then there exists a subset X of A with fewer than m elements, such that no member of G agrees with φ on X. We can obviously choose the set X in such a way that it has at least two elements. Arranging the members of X into a one-to-one sequence x, we therefore have $x \epsilon K$. Obviously $x \epsilon Gx$, but on the other hand $\varphi x \notin Gx$. Consequently

$$F_x(\varphi x) = \varphi(x_1) \neq \varphi(x_0) = \varphi\big(F_x(x)\big),$$

and φ is therefore not an automorphism of $\boldsymbol{A}$.

The cases $m = 0$ and $m = 1$ are trivial. Each of the conditions $\beta_0(G)$ and $\beta_1(G)$ implies that G is the group of all permutations of $\boldsymbol{A}$, and we may therefore take for our algebra $\boldsymbol{A}$ the algebra with base set $\boldsymbol{A}$ and no operations.

THEOREM 3. *For every infinite cardinal m, the conditions $\alpha_m(G)$ and $\beta_m(G)$ are equivalent.*

Proof. By Theorems 1 and 2.

The three counterexamples that follow supplement the above results.

EXAMPLE 1. *$\beta_2(G)$ does not imply $\alpha_2(G)$.*

Let A be the union of two disjoint sets B and C, each with at least three elements, and let G be the group of all those permutations of A that map B onto B and C onto C. Clearly $\beta_2(G)$ holds. We claim that the only unary operation F that is preserved by every member of G is the identity map. In fact, if $F(x) = y \neq x$, then there exist $\sigma, \sigma' \epsilon G$ such that $\sigma(x) = \sigma'(x)$ but $\sigma(y) \neq \sigma'(y)$. If F were preserved by both σ and σ', then we would have

$$\sigma(y) = F\big(\sigma(x)\big) = F\big(\sigma'(x)\big) = \sigma'(y),$$

a contradiction. It readily follows that $\alpha_2(G)$ fails.

EXAMPLE 2. *If m is a non-zero finite cardinal, then $\alpha_m(G)$ does not imply $\beta_m(G)$.*

Let A be a set with $m+1$ elements, and let G be the alternating group on A. Given an $(m-1)$-element subset X of A, every one-to-one mapping of X into A can be extended in exactly two ways to a permutation of A, and one of these permutations is even, the other odd. Therefore, every permutation of A agrees on X with some member of G. This shows that $\beta_m(G)$ fails.

To prove that $\alpha_m(G)$ holds, we construct a single operation F of rank $m-1$ over A, such that the group of all automorphisms of the algebra $\mathbf{A} = \langle A, F\rangle$ is G. Arrange the elements of A into a one-to-one sequence $x = \langle x_0, x_1, \ldots, x_m\rangle$. If the $(m-1)$-termed sequence y of elements of A is not one-to-one, then we let $F(y) = y_0$. On the other hand, if y is one-to-one, then there exists exactly one member σ of G that maps x_i onto y_i for $i = 0, 1, \ldots, m-2$, and we let $F(y) = \sigma(x_{m-1})$.

If y is not one-to-one, then obviously $F(\varphi y) = \varphi\big(F(y)\big)$ for every permutation φ of A. Consider now the case when y is one-to-one, and let σ be the member of G that maps x_i onto y_i for $i = 0, 1, \ldots, m-2$. If the permutation φ of A is even, then the member σ' of G that maps x_i onto $\varphi(y_i)$ for $i = 0, 1, \ldots, m-2$ is equal to $\varphi\sigma$, and therefore

$$F(\varphi y) = \varphi\sigma(x_{m-1}) = \varphi\big(F(y)\big).$$

On the other hand, if φ is odd, then $\sigma' \neq \varphi\sigma$. Since two distinct permutations must differ in at least two places, we must have $\varphi\big(\sigma(x_{m-1})\big) \neq \sigma'(x_{m-1})$. Therefore

$$F(\varphi y) = \sigma'(x_{m-1}) \neq \varphi\sigma(x_{m-1}) = \varphi\big(F(y)\big).$$

This shows that the automorphism group of $\mathbf{A}$ is G.

Example 3. *If m is a non-zero finite cardinal, then $\beta_{m+1}(G)$ does not imply $\alpha_m(G)$.*

Proof. First consider the case $m > 1$. Let A be the union of disjoint sets $B_0, B_1, \ldots, B_m$, all with the same number n of elements, where $n \geqslant m+2$. Let G' be the group of all permutations φ of A such that φ maps each of the sets B_i onto a set B_j. Thus each $\varphi \in G'$ induces a permutation φ^* of the family $\mathfrak{B} = \{B_0, B_1, \ldots, B_m\}$. We let G be the set of all $\varphi \in G'$ such that φ^* is even.

Consider any permutation φ of A that satisfies the hypothesis of $\beta_{m+1}(G)$. If $u, v \in B_i$, then φ agrees on $\{u, v\}$ with some member of G, and therefore $\varphi(u)$ and $\varphi(v)$ belong to the same member of $\mathfrak{B}$. Thus $\varphi \in G'$. Furthermore, if we pick $x_i \in B_i$ for $i = 0, 1, \ldots, m-1$, then φ agrees on the set $\{x_0, x_1, \ldots, x_{m-1}\}$ with some member σ of G. The induced maps φ^* and σ^* agree on $m-1$ of the members of $\mathfrak{B}$, and must therefore be equal. Consequently φ^* is even, so that $\varphi \in G$.

In order to prove that $\alpha_m(G)$ fails, we shall show that if an operation F of rank $r < m$ is preserved by all the members of G, then it is preserved by every member of G'. Observe first that if x is any r-termed sequence of elements of A, then $F(x)$ must be one of the terms x_p, for otherwise we can find a permutation in G that moves $F(x)$ but leaves all the terms x_i fixed. Given $\varphi \in G'$, it is also easy to see that $\varphi x = \sigma x$ for some $\sigma \in G$, and from this it follows that

$$F(\varphi x) = F(\sigma x) = \sigma\big(F(x)\big) = \sigma(x_p) = \varphi(x_p) = \varphi\big(F(x)\big).$$

Regarding the case $m = 1$, we observe that $\beta_2(G)$ holds iff A is partitioned into sets B_i, with G consisting of precisely those permutations of A that map each B_i onto itself. On the other hand, $\alpha_1(G)$ means that G is the set of all automorphisms of an algebra with a set U of distinguished element, and with no operations of positive rank; i.e., G is the group of all those permutations of A that leave each member of U fixed. From this it is clear that $\beta_2(G)$ does not imply $\alpha_1(G)$.

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