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A REMARK ON INDEPENDENCE IN PROJECTIVE SPACES

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1. Many notions of independence can be treated as the general algebraic independence in suitably defined abstract algebras (compare [4]), but there are some exceptions (see [4] and [3]).

E. Marczewski has raised the following problem (see [5], problem P 522):

Let P_n be a projective n-dimensional space. Does there exist an algebra $(P_n, \boldsymbol{F})$ such that

(α) *a set $I \subseteq P_n$ is linearly independent in P_n if and only if I is independent in the algebra $(P_n; \boldsymbol{F})$.*

In this note I deduce a negative solution of this problem from a representation theorem of Urbanik [7] under an additional but natural assumption that

(β) *every subalgebra of $(P_n; \boldsymbol{F})$ is a subspace of P_n.*

Let n and $\boldsymbol{F}$ be fixed and let $X \subseteq P_n$. By $P(X)$ I denote the subspace of P_n generated by set X and by $C(X)$ the subalgebra of $(P_n; \boldsymbol{F})$ generated by this set.

For definitions connected with algebric independence see [6].

I am very grateful to Professor Marczewski for his detailed remarks utilized in this note.

2. THEOREM. *If $n \geqslant 2$, then there exists no algebra $(P_n; \boldsymbol{F})$ satisfying* (α) *and* (β).

Let us consider the condition

(β^*) *a subset S of P_n is a subalgebra of $(P_n; \boldsymbol{F})$ if and only if S is a subspace of P_n (or, in other words, $P(S) = C(S)$ for every $S \subseteq P_n$).*

It is stronger then (β).

I shall prove two lemmas.

LEMMA 1. *If $(P_n; \boldsymbol{F})$ satisfies* (α) *and* (β), *then it satisfies* (β^*).

Proof. It follows from (β) that $P(E) \subseteq C(E)$ for every independent set $E \subseteq P$. I shall show that $C(E) \subseteq P(E)$. Let $p \in C(E)$. The set $E \cup \{p\}$

is dependent in algebra $(P_n; \boldsymbol{F})$ and, consequently, it is linearly dependent in P_n, whence $p \in P(E)$. So we have $C(E) = P(E)$, and since for every $X \subseteq P_n$ there exists an independent set E such that $P(X) = P(E) = C(E)$, we have that every subspace of P_n is a subalgebra of $(P_n; \boldsymbol{F})$ which together with (β) gives (β^*), q. e. d.

LEMMA 2. *If* $n \geqslant 2$, *then there exists no algebra* $(P_n; \boldsymbol{F})$ *satisfying* (α) *and* (β^*).

Proof. Suppose that there exists an algebra $(P_n; \boldsymbol{F})$ satisfying (α) and (β^*). It follows from (α) and from the properties of linear independence that for every set E independent in $(P_n; \boldsymbol{F})$ the set $E \cup \{p\}$ is independent in $(P_n; \boldsymbol{F})$ if and only if $p \notin C(E)$. Hence $(P_n; \boldsymbol{F})$ is ϑ^*-algebra (see [7]).

Since every two-element set in a projective space is linearly independent, then every unary algebraic operation in the algebra $(P_n; \boldsymbol{F})$ is trivial. On the other hand, however, every projective plane contains a 3-element dependent set, hence there is in the algebra $(P_n; \boldsymbol{F})$ an algebraic operation essentially depending on two or three variables.

Therefore in view of the Urbanik's representation theorem for v^*-algebras (see [7]) $(P_n; \boldsymbol{F})$ is an affine space, that is

$$(P_n; \boldsymbol{F}) = \Big(K^n; \sum_{i=1}^{m} a_i x_i\Big),$$

where K is a field,

$$a_i \in K, \qquad \sum_{i=1}^{m} a_i = 1, \qquad m = 1, 2, \ldots$$

and K^m denotes the m-th cartesian power of the set K.

The idea of the subsequent part of the proof is this. From what we have said it follows that the family of all subalgebras of the algebra $(P_n; \boldsymbol{F})$ is the family of all subspaces of an affine space and at the same time it is the family of all subspaces of a projective space. This is, however, impossible and so we have a contradiction.

To be more precise, let us consider a 3-element subset T of P_n, independent in $(P_n; \boldsymbol{F})$. It is easily seen that for the projective subspace $P(T)$ the subalgebra $C(T)$ satisfies conditions (α) and (β^*). The subalgebra $C(T)$ is isomorphic with the algebra $(K^2; \sum a_i x_i)$, where $\sum a_i = 1$ and the part of straight lines in projective plane $P(T)$ is played by subalgebras generated by 2-element sets. But the subalgebra $C(T)$ contains two disjoint such subalgebras, for example the set $\{(x, y): x = 0, y \in K$ and $\{(x, y): x = 1, y \in K\}$, which yields a contradiction, because every two straight lines in a projective plane have a point in common, q. e. d.

Now our theorem follows from Lemmas 1 and 2.

Remark. Every algebra $(P_1, \boldsymbol{F})$ in which every 2-element set is independent satisfies (α). If every 2-element set is a basis, then $(P_1; \boldsymbol{F})$ satisfies (β^*). A characterization of such algebras is to be found in [2].

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