SUMMABILITY OF PURE EXTENSIONS
OF RELATIONAL STRUCTURES

BY

B. WĘGLORZ AND A. WOJCIECHOWSKA (WROCŁAW)

In this paper we always consider algebraic structures of a fixed similarity type and a first order language with identity corresponding to this type.

Let Σ be an arbitrary set of formulas of our language, and $\mathfrak{A}$ an arbitrary structure. Then by $\Sigma(\mathfrak{A})$ we denote the corresponding set of all formulas with constants in $\mathfrak{A}$, i.e. of all formulas which can be obtained from Σ by substituting some elements of $\mathfrak{A}$ for some free variables. As to the notation and general suppositions which are not explained here, see [9]. For a formula φ having only $x_1, \ldots, x_n$ as free variables, the symbol $\mathrm{H}(\varphi)$ denotes the sentence $\mathrm{H} x_1 \ldots x_n \varphi$. We recall that a *sentence* is a formula with constants without free variables.

Let $\mathfrak{A} \subseteq \mathfrak{B}$. Then $\mathfrak{A}$ is *downward Σ-pure* (*upward Σ-pure*) in $\mathfrak{B}$ if, for every formula $\varphi \in \Sigma(\mathfrak{A})$, $\mathfrak{B} \models \mathrm{H}(\varphi)$ implies that $\mathfrak{A} \models \mathrm{H}(\varphi)$ ($\mathfrak{A} \models \mathrm{H}(\varphi)$ implies that $\mathfrak{B} \models \mathrm{H}(\varphi)$, respectively).

If $\mathfrak{A}$ is downward and upward Σ-pure in $\mathfrak{B}$, then we say that $\mathfrak{A}$ is *Σ-pure* in $\mathfrak{B}$. If Σ is the set of all conjunctions of atomic formulas, all positive formulas, all Horn [1] formulas or all formulas, then for *Σ-pure* we will say *pure, positively pure, Horn-pure* or *elementarily pure*, respectively. Moreover, each of those qualifications may be acompanied by the adverb *downward* or *upward* whose meaning was already defined.

For various sets Σ, the notions of downward Σ-purity, upward Σ-purity and Σ-purity were already studied. E.g. purity was already defined and used in [9], and it generalizes a well known notion from the theory of Abelian groups. Sometimes those three notions coincide as we will see in the following examples.

(1) $\mathfrak{A}$ is downward pure in $\mathfrak{B}$ if and only if $\mathfrak{A}$ is pure in $\mathfrak{B}$, and $\mathfrak{A}$ is upward pure in $\mathfrak{B}$ if and only if $\mathfrak{A} \subseteq \mathfrak{B}$.

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[1] For the definition of Horn formulas and atomic Horn formulas see e.g. [2].

(2) $\mathfrak{A}$ is downward (upward) elementarily pure in $\mathfrak{B}$ if and only if $\mathfrak{A}$ is an elementary substructure of $\mathfrak{B}$ (see Lemmas 4 and 5 below).

(3) If Σ consists of all sentences, then the downward Σ-purity, upward Σ-purity or Σ-purity of $\mathfrak{A}$ in $\mathfrak{B}$ are all equivalent to $\mathfrak{A} \subseteq \mathfrak{B}$ and $Th(\mathfrak{A}) = Th(\mathfrak{B})$.

(4) If Σ consists of all universal sentences, then $\mathfrak{A}$ is downward Σ-pure in $\mathfrak{B}$ if and only if $\mathfrak{A} \subseteq \mathfrak{B}$.

(5) The diagonal substructure of a direct power of a structure is upward Horn-pure in this direct power. On the other hand, the diagonal substructure is not always downward Horn-pure in this direct power.

(6) The notions of downward positive purity, upward positive purity and positive purity are all different.

PROPOSITION 1. *Let $\mathfrak{A} \subseteq \mathfrak{B}$. Then $\mathfrak{A}$ is downward (upward) Σ-pure in $\mathfrak{B}$ if and only if there is a structure $\mathfrak{C}$ such that $\mathfrak{A}$ is downward (upward) Σ-pure in $\mathfrak{C}$ and $\mathfrak{B}$ is upward (downward) Σ-pure in $\mathfrak{C}$.*

Proof. If $\mathfrak{A}$ is downward Σ-pure (upward Σ-pure) in $\mathfrak{B}$, then it suffices to put $\mathfrak{B} = \mathfrak{C}$.

Conversely, suppose that a structure $\mathfrak{C}$ satisfying the condition of Proposition 1 exists. Let $\varphi \epsilon \Sigma(\mathfrak{A}) \subseteq \Sigma(\mathfrak{B})$ and suppose that $\mathfrak{B} \models \mathfrak{H}(\varphi)$. Then we have $\mathfrak{C} \models \mathfrak{H}(\varphi)$ and also $\mathfrak{A} \models E(\varphi)$, since $\mathfrak{A}$ is downward Σ-pure in $\mathfrak{C}$.

In the case when $\mathfrak{A}$ is upward Σ-pure in $\mathfrak{C}$ and $\mathfrak{B}$ is downward Σ-pure in $\mathfrak{C}$, the proof is analogous.

PROPOSITION 2. *Let $\mathfrak{A} \subseteq \mathfrak{B}$. Then $\mathfrak{A}$ is Σ-pure in $\mathfrak{B}$ if and only if there is a structure $\mathfrak{C} \supseteq \mathfrak{B}$ such that $\mathfrak{A}$ and $\mathfrak{B}$ are Σ-pure in $\mathfrak{C}$.*

Proof. By Proposition 1.

PROPOSITION 3. *Let $\mathfrak{A} \subseteq \mathfrak{B} \subseteq \mathfrak{C}$, where $\mathfrak{A}$ is downward (upward) Σ-pure in $\mathfrak{B}$ and $\mathfrak{B}$ is downward (upward) Σ-pure in $\mathfrak{C}$. Then $\mathfrak{A}$ is downward (upward) Σ-pure in $\mathfrak{C}$.*

Proof. Obvious.

Now we give two obvious but essential lemmas.

LEMMA 4. *If $\mathfrak{A}$ is downward Σ-pure in $\mathfrak{B}$, then for each formula $\varphi \epsilon \Sigma(\mathfrak{A})$ and each sequence $\boldsymbol{a} \epsilon A^{\omega}$*

$$\mathfrak{B} \models \varphi[\boldsymbol{a}] \text{ implies that } \mathfrak{A} \models \varphi[\boldsymbol{a}].$$

LEMMA 5. *If $\mathfrak{A}$ is upward Σ-pure in $\mathfrak{B}$, then for each formula $\varphi \epsilon \Sigma(\mathfrak{A})$ and each sequence $\boldsymbol{a} \epsilon A^{\omega}$*

$$\mathfrak{A} \models \varphi[\boldsymbol{a}] \text{ implies that } \mathfrak{B} \models \varphi[\boldsymbol{a}].$$

COROLLARY 6. *If $\mathfrak{A}$ is Σ-pure in $\mathfrak{B}$, then for each formula $\varphi \epsilon \Sigma(\mathfrak{A})$ and each sequence $\boldsymbol{a} \epsilon A^{\omega}$*

$$\mathfrak{A} \models \varphi[\boldsymbol{a}] \text{ if and only if } \mathfrak{B} \models \varphi[\boldsymbol{a}].$$

We say that the Σ-purity (downward Σ-purity) [upward Σ-purity] is *summable*, if each ascending chain of structures

$$\mathfrak{A}_0 \subseteq \mathfrak{A}_1 \subseteq \ldots \subseteq \mathfrak{A}_n \subseteq \ldots \quad (n < \omega),$$

such that $\mathfrak{A}_n$ is Σ-pure (downward Σ-pure) [upward Σ-pure] in $\mathfrak{A}_{n+1}$ for each $n < \omega$, is such that $\mathfrak{A}_n$ is Σ-pure (downward Σ-pure) [upward Σ-pure] in $\bigcup_{n<\omega} \mathfrak{A}_n$ for each $n < \omega$.

It is easy to see that the downward Σ-purity in cases (1), (2) and (4) is summable and that the Σ-purity in the case (3) is not summable. There is of course the classical result of Tarski and Vaught [7] (see also our Corollary 11) that elementary purity is summable. Some other papers on chains and sums of such chains (see e.g. [1], [3] and [4]) concern such a Σ-purity which is summable. We do not know any characterization of such sets Σ for which Σ-purity or downward Σ-purity or upward Σ-purity is summable. The following theorem shows how from a given set Σ such that the Σ-purity is summable we can obtain larger sets having the same property.

For any set Σ of formulas $\Sigma_\wedge$ denotes the set consisting of all finite conjunctions of formulas from Σ;

$\Sigma_\vee$ denotes the set consisting of all finite disjunctions of formulas from Σ;

$\Sigma_\exists$ denotes the set consisting of all formulas of the form $\exists x \varphi$ for $\varphi \in \Sigma$, x being one of the free variables of φ;

$\Sigma_\forall$ denotes the set consisting of all formulas of the form $\forall x \varphi$ for $\varphi \in \Sigma$, x being one of the free variables of φ;

$\Sigma_\neg$ denotes the set consisting of all negations of formulas of Σ.

THEOREM 7. *If Σ is a set of formulas for which the downward Σ-purity (upward Σ-purity) is summable, then each of the sets $\Sigma_\wedge, \Sigma_\vee, \Sigma_\exists, \Sigma_\forall$, has the same property.*

Proof. Let $\mathfrak{A}_0 \subseteq \mathfrak{A}_1 \subseteq \ldots \subseteq \mathfrak{A}_n \subseteq \ldots (n < \omega)$, be a chain of structures such that for each $n < \omega$, $\mathfrak{A}_n$ is downward Σ_*-pure (upward Σ_*-pure) in $\mathfrak{A}_{n+1}$, where Σ_* denotes any of the sets $\Sigma_\wedge, \Sigma_\vee, \Sigma_\exists, \Sigma_\forall$, and $\mathfrak{B} = \bigcup_{n<\omega} \mathfrak{A}_n$.

Downward $\Sigma_\wedge$-purity. Suppose that a formula of the form $\varphi \wedge \varphi$, where $\varphi, \psi \in \Sigma(\mathfrak{A}_k)$, is satisfiable in $\mathfrak{B}$. Let $x_1, \ldots, x_n$ be the free variables of $\varphi \wedge \psi$. This means that there is a sequence $\langle b_1, \ldots, b_n \rangle \in B^n$ such that $\mathfrak{B} \models (\varphi \wedge \psi)[b_n, \ldots, b_n]$, so that, consequently, $\mathfrak{B} \models \varphi[b_1, \ldots, b_n]$ and $\mathfrak{B} \models \psi[b_1, \ldots, b_n]$. Let $\mathfrak{A}_m$ be a structure such that $\mathfrak{A}_k \subseteq \mathfrak{A}_m$ and $b_1, \ldots, b_n \in A_m$. Since the downward Σ-purity is summable by hypothesis, we have $\mathfrak{A}_m \models \varphi[b_1, \ldots, b_n]$ and $\mathfrak{A}_m \models \psi[b_1, \ldots, b_n]$, by Lemma 4. Thus $\mathfrak{A}_m \models (\varphi \wedge \psi)[b_1, \ldots, b_n]$ and $\mathfrak{A}_m \models \exists(\varphi \wedge \psi)$. Consequently, $\mathfrak{A}_k \models \exists(\varphi \wedge \psi)$, since $\mathfrak{A}_k$ is downward $\Sigma_\wedge$-pure in $\mathfrak{A}_m$ (by Proposition 3).

Upward $\Sigma_\wedge$-purity. Suppose that $\varphi, \psi \in \Sigma(\mathfrak{A}_k)$ and $\mathfrak{A}_k \models \mathfrak{H}(\varphi \wedge \psi)$. By an analogous reasoning using Lemma 5 instead of Lemma 4, we obtain $\mathfrak{B} \models \mathfrak{H}(\varphi \wedge \psi)$.

For the *downward $\Sigma_\vee$-purity* the proof is easy.

Upward $\Sigma_\vee$-purity. Suppose that $\varphi, \psi \in \Sigma(\mathfrak{A}_k)$ and $\mathfrak{A}_k \models \mathfrak{H}(\varphi \vee \psi)$. Thus there is a sequence $\boldsymbol{a} \in A_k^\omega$ such that $\mathfrak{A}_k \models (\varphi \vee \psi)[\boldsymbol{a}]$, whence $\mathfrak{A}_k \models \varphi[\boldsymbol{a}]$ or $\mathfrak{A}_k \models \psi[\boldsymbol{a}]$; say $\mathfrak{A}_k \models \varphi[\boldsymbol{a}]$. Since $\mathfrak{A}_k$ is upward Σ-pure in $\mathfrak{A}_m$ for each $m \geqslant k$, $\mathfrak{A}_m \models \varphi[\boldsymbol{a}]$. Now, using the summability of the upward Σ-purity, Lemma 5, and the fact that $\mathfrak{B} = \bigcup_{k \leqslant n < \omega} \mathfrak{A}_n$, we have $\mathfrak{B} \models \varphi[\boldsymbol{a}]$. So $\mathfrak{B} \models \mathfrak{H}(\varphi \vee \psi)$.

For the *downward and upward $\Sigma_{\mathfrak{H}}$-purity* the proofs are easy.

Downward $\Sigma_\forall$-purity. Suppose that a formula of the form $\forall x_n \varphi$ is satisfiable in $\mathfrak{B}$, for $\varphi \in \Sigma(\mathfrak{A}_k)$. This means that there is a sequence $\boldsymbol{b} = \langle b_0, \ldots, b_{n-1}, b_n, b_{n+1}, \ldots \rangle \in B^\omega$ such that, if we substitute b_n for an arbitrary $b_n' \in B$, we get that the sequence $\langle b_0, \ldots, b_{n-1}, b_n', b_{n+1}, \ldots \rangle$ satisfies φ in $\mathfrak{B}$. Let $x_0, \ldots, x_m$ be all the free variables of $\forall x_n \varphi$. Since $\mathfrak{B} = \bigcup_{n < \omega} \mathfrak{A}_n$, there is an $r < \omega$ such that $\mathfrak{A}_r \supseteq \mathfrak{A}_k$ and $b_0, \ldots, b_m \in A_r$. By the previous remark, there is a sequence $\boldsymbol{b}' \in A_r^\omega$ which satisfies $\forall x_n \varphi$ in $\mathfrak{B}$. Such a $\boldsymbol{b}'$ can be obtained by putting $b_0' = b_0, \ldots, b_m' = b_m$ and $b_s' \in A_r$ for $s > m$. Then for an arbitrary element $b_n'' \in B$ the sequence obtained from $\boldsymbol{b}'$ by the substitution of $b_n'' \in B$ on the n-th place, satisfies φ in $\mathfrak{B}$, in particular it is so for all $b_n'' \in A_r$. Thus, using the summability of the downward Σ-purity and Lemma 4, we see that every sequence obtained from $\boldsymbol{b}'$ by substituting an arbitrary element $b_n'' \in A_r$ on the n-th place satisfies φ in $\mathfrak{A}_r$. Thus $\boldsymbol{b}'$ satisfies $\forall x_n \varphi$ in $\mathfrak{A}_r$. Since $\mathfrak{A}_r \supseteq \mathfrak{A}_k$, we obtain that $\forall x_n \varphi$ is satisfiable in $\mathfrak{A}_k$.

Upward $\Sigma_\forall$-purity. Let $\varphi \in \Sigma(\mathfrak{A}_k)$ and let the formula $\forall x_n \varphi$ be satisfiable in $\mathfrak{A}_k$. This means that there is a sequence $\boldsymbol{a} = \langle a_0, \ldots, a_{n-1}, a_n, a_{n+1}, \ldots \rangle \in A_k^\omega$ such that for an arbitrary $a_n' \in A_k$ a sequence obtained from $\boldsymbol{a}$ by substituting a_n' on the n-th place satisfies φ in $\mathfrak{A}_k$. Thus, using summability of the upward Σ-purity and Lemma 5, every one of such sequences satisfies φ in $\mathfrak{B}$. Suppose to the contrary that the sequence $\boldsymbol{a}$ does not satisfy $\forall x_n \varphi$ in $\mathfrak{B}$. This means that there is an element $b_n \in B$ such that the sequence $\boldsymbol{b} = \langle a_0, \ldots, a_{n-1}, b_n, a_{n+1}, \ldots \rangle$ does not satisfy $x_n \varphi$ in $\mathfrak{B}$. Since $\mathfrak{B} = \bigcup_{n < \omega} \mathfrak{A}_n$, there is a structure $\mathfrak{A}_r \supseteq \mathfrak{A}_k$ such that $b_n \in A_r$. Using Lemma 5 and the summability of the upward Σ-purity we have $\mathfrak{A}_r \models \neg \varphi[\boldsymbol{b}]$. Now, let us observe that if we replace in φ all the free variables different from x_n by the corresponding elements from $\boldsymbol{a}$, then we obtain a formula $\psi \in \Sigma(\mathfrak{A}_k)$ such that $\mathfrak{A}_k \models \forall x_n \psi$. Since $\mathfrak{A}_k$ is upward $\Sigma_\forall$-pure in $\mathfrak{A}_r$, we have $\mathfrak{A}_r \models \forall x_n \psi$, but this is impossible since $\mathfrak{A}_r \models \neg \varphi[\boldsymbol{b}]$ implies $\mathfrak{A}_r \models \neg \forall x_n \psi$. This contradiction finishes the proof.

We do not know if for any set Σ of formulas, the downward Σ-purity is summable if and only if the upward $\Sigma_\neg$-purity is summable. We only can prove the following

THEOREM 8. *Let Σ be a set of formulas for which the downward Σ-purity (upward Σ-purity) is summable, and let $\mathfrak{A}_n$ be downward Σ-pure (upward Σ-pure) in $\mathfrak{A}_{n+1}$ for all $n < \omega$. If, moreover, $\mathfrak{A}_n$ is upward $\Sigma_\neg$-pure (downward $\Sigma_\neg$-pure) in $\mathfrak{A}_{n+1}$ for $n < \omega$, then each $\mathfrak{A}_n$ is upward $\Sigma_\neg$-pure (downward $\Sigma_\neg$-pure) in $\mathfrak{B} = \bigcup_{n<\omega} \mathfrak{A}_n$.*

Proof. Suppose that the downward Σ-purity is summable and let $\neg\varphi$ be satisfiable in $\mathfrak{A}_k$, where $\varphi \epsilon \Sigma(\mathfrak{A}_k)$. This means that there is a sequence $\boldsymbol{a} \epsilon A_k^\omega$ such that $\mathfrak{A}_k \models \neg\varphi[\boldsymbol{a}]$. Using Lemma 4 and the fact that the downward Σ-purity is summable we obtain $\mathfrak{B} \models \neg\varphi[\boldsymbol{a}]$, whence $\neg\varphi$ is satisfiable in $\mathfrak{B}$.

Now, let us suppose that the upward Σ-purity is summable. Let $\varphi \epsilon \Sigma_\neg(\mathfrak{A}_k)$ be satisfiable in $\mathfrak{B}$. This means that there is a sequence $\boldsymbol{b} \epsilon B^\omega$ such that $B \models \varphi[\boldsymbol{b}]$. Let $x_0, \ldots, x_n$ be all the free variables of φ. Thus there is a structure $\mathfrak{A}_r \supseteq \mathfrak{A}_k$ such that $b_0, \ldots, b_n \epsilon A_r$ and a sequence $\boldsymbol{b}' \epsilon A_r^\omega$ such that $\mathfrak{B} \models \varphi[\boldsymbol{b}']$. Since the upward Σ-purity is summable, $\mathfrak{A}_r$ is upward Σ-pure in $\mathfrak{B}$, whence, by Lemma 5, $\mathfrak{A}_r \models \varphi[\boldsymbol{b}']$. Thus φ is satisfiable in $\mathfrak{A}_r$ and hence also in $\mathfrak{A}_k$.

COROLLARY 9. *If the Σ-purity is summable, then the $(\Sigma \cup \Sigma_\neg)$-purity is also summable.*

Corollary 9 follows immediately from Theorem 8 and the following simple

LEMMA 10. *If for every $i \epsilon I$ the upward (downward) Σ_i-purity is summable, then the upward (downward) $(\bigcup_{i \epsilon I} \Sigma_i)$-purity is also summable.*

Lemma 10 will be also used to obtain

COROLLARY 11. (i) (Tarski-Vaught) *If, for each $n < \omega$, $\mathfrak{A}_n$ is an elementary substructure of $\mathfrak{A}_{n+1}$, then each $\mathfrak{A}_n$ is an elementary substructure of $\bigcup_{n<\omega} \mathfrak{A}_n$.*

(ii) *If, for each $n < \omega$, $\mathfrak{A}_n$ is downward (upward) positively pure in $\mathfrak{A}_{n+1}$, then each $\mathfrak{A}_n$ is downward (upward) positively pure in $\bigcup_{n<\omega} \mathfrak{A}_n$.*

(iii) *If, for each $n < \omega$, $\mathfrak{A}_n$ is downward (upward) Horn-pure in $\mathfrak{A}_{n+1}$, then each $\mathfrak{A}_n$ is downward (upward) Horn-pure in $\bigcup_{n<\omega} \mathfrak{A}_n$.*

Proof. (i) and (ii) follow immediately from Theorems 7, 8, and Lemma 10. By Theorem 7 and Lemma 10, to prove (iii) it suffices to show that the downward (upward) Σ-purity is summable, where Σ is the set of basic Horn formulas. But this is almost trivial.

Since now on, we always assume that Σ is such that the downward and upward Σ-purity is summable, which, ·by the previous results, is a natural assumption.

A class $\mathscr{K}$ of algebraic structures is said to be Σ-*closed* (*downward* Σ-*closed*) [*upward* Σ-*closed*] if for every $\mathfrak{A}, \mathfrak{B} \epsilon \mathscr{K}$ the inclusion $\mathfrak{A} \subseteq \mathfrak{B}$ implies that $\mathfrak{A}$ is Σ-pure (downward Σ-pure), [upward Σ-pure] in $\mathfrak{B}$. If Σ consists of all formulas for "Σ-*closed*" we sill say "*elementarily closed*".

Elementarily closed elementary classes are especially important because those are exactly the classes defined by a model-complete theory (see [6]).

PROPOSITION 12. (i) $\mathscr{K}$ *is downward* Σ-*closed if and only if for each* $\mathfrak{A} \epsilon \mathscr{K}$ *and* $\varphi \epsilon \Sigma(\mathfrak{A})$ *either* φ *is satisfiable in* $\mathfrak{A}$ *or there is no extension of* $\mathfrak{A}$ *belonging to* $\mathscr{K}$ *in which* φ *is satisfiable.*

(ii) $\mathscr{K}$ *is upward* Σ-*closed if and only if for each* $\mathfrak{A} \epsilon \mathscr{K}$ *and* $\varphi \epsilon \Sigma(\mathfrak{A})$ *either* φ *is not satisfiable in* $\mathfrak{A}$ *or, for each extension* $\mathfrak{B} \epsilon \mathscr{K}$ *of* $\mathfrak{A}$, φ *is satisfiable in* $\mathfrak{B}$.

THEOREM 13. *Let* $\mathscr{K}$ *be a class of structures such that, for every* $\mathfrak{A}, \mathfrak{B} \epsilon \mathscr{K}$, *if* $\mathfrak{A} \subseteq \mathfrak{B}$, *then there exists a* $\mathfrak{C} \epsilon \mathscr{K}$ *such that* $\mathfrak{B} \subseteq \mathfrak{C}$ *and* $\mathfrak{A}$ *is* Σ-*pure in* $\mathfrak{C}$. *Then* $\mathfrak{A}$ *is* Σ-*pure in* $\mathfrak{B}$.

Proof. We can construct by induction a sequence of structures $\langle \mathfrak{A}_n \rangle_{n<\omega}$ such that $\mathfrak{A}_0 = \mathfrak{A}, \mathfrak{A}_1 = \mathfrak{B}$ and, for each $n < \omega$, $\mathfrak{A}_n \epsilon \mathscr{K}$, $\mathfrak{A}_n \subseteq \mathfrak{A}_{n+1}$ and $\mathfrak{A}_n$ is Σ-pure in $\mathfrak{A}_{n+2}$.

Putting

$$\mathfrak{A}^* = \bigcup_{n<\omega} \mathfrak{A}_n = \bigcup_{n<\omega} \mathfrak{A}_{2n} = \bigcup_{n<\omega} \mathfrak{A}_{2n+1}$$

we obtain a structure $\mathfrak{A}^*$ such that $\mathfrak{A}$ and $\mathfrak{B}$ are Σ-pure in $\mathfrak{A}^*$. Thus Theorem 13 follows from Proposition 2.

Remark. The structure $\mathfrak{A}$ constructed in this proof need not belong to $\mathscr{K}$.

COROLLARY 14. $\mathscr{K}$ *is* Σ-*closed if and only if for every* $\mathfrak{A}, \mathfrak{B} \epsilon \mathscr{K}$, *where* $\mathfrak{A} \subseteq \mathfrak{B}$, *there is a* $\mathfrak{C} \epsilon \mathscr{K}$ *such that* $\mathfrak{B} \subseteq \mathfrak{C}$ *and* $\mathfrak{A}$ *is* Σ-*pure in* $\mathfrak{C}$.

Proof. The necessity of this condition follows by putting $\mathfrak{C} = \mathfrak{B}$; the sufficiency by Theorem 13.

THEOREM 15. *Let* $\mathscr{K}$ *be a class of structures such that, for every* $\mathfrak{A}$, $\mathfrak{B} \epsilon \mathscr{K}$, *if* $\mathfrak{A} \subseteq \mathfrak{B}$, *then there are* $\mathfrak{C}$ *and* $\mathfrak{D}$ *from* $\mathscr{K}$ *such that* $\mathfrak{A} \subseteq \mathfrak{B} \subseteq \mathfrak{C} \subseteq \mathfrak{D}$, $\mathfrak{A}$ *is downward (upward)* Σ-*pure in* $\mathfrak{C}$ *and* $\mathfrak{B}$ *is upward (downward)* Σ-*pure in* $\mathfrak{D}$. *Thus* $\mathfrak{A}$ *is downward (upward)* Σ-*pure in* $\mathfrak{B}$.

Proof. We can construct by induction a sequence of structures $\langle \mathfrak{A}_n \rangle_{n<\omega}$ such that $\mathfrak{A}_0 = \mathfrak{A}, \mathfrak{A}_1 = \mathfrak{B}$ and, for each $n < \omega$, $\mathfrak{A}_n \epsilon \mathscr{K}$, $\mathfrak{A}_n \subseteq \mathfrak{A}_{n+1}$, $\mathfrak{A}_{2n}$ is downward (upward) Σ-pure in $\mathfrak{A}_{2n+2}$, $\mathfrak{A}_{2n+1}$ is upward (downward) Σ-pure in $\mathfrak{A}_{2n+3}$.

Putting

$$\mathfrak{A}^* = \bigcup_{n<\omega} \mathfrak{A}_n = \bigcup_{n<\omega} \mathfrak{A}_{2n} = \bigcup_{n<\omega} \mathfrak{A}_{2n+1},$$

we obtain a structure $\mathfrak{A}^*$ such that $\mathfrak{A}$ and $\mathfrak{B}$ satisfy the assumption of Proposition 1. Thus Theorem 15 follows from Proposition 1.

COROLLARY 16. *$\mathcal{K}$ is downward (upward) Σ-closed if and only if for every $\mathfrak{A}, \mathfrak{B} \in \mathcal{K}$, where $\mathfrak{A} \subseteq \mathfrak{B}$, there are $\mathfrak{C}, \mathfrak{D} \in \mathcal{K}$ such that $\mathfrak{B} \subseteq \mathfrak{C} \subseteq \mathfrak{D}$ and $\mathfrak{A}$ is downward (upward) Σ-pure in $\mathfrak{C}$ and $\mathfrak{B}$ is upward (downward) Σ-pure in $\mathfrak{D}$.*

Proof. The necessity of this condition follows by putting $\mathfrak{C} = \mathfrak{D} = \mathfrak{B}$; the sufficiency by Theorem 15.

Finally, we add some facts relating purity and atomic compactness (see [9]). We will discuss the possibility of imbedding a given structure in an atomic compact one and solve a problem stated in [10] concerning purity of such imbeddings.

PROPOSITION 17. *If $\mathfrak{A}_0 \subseteq \mathfrak{A}_1 \subseteq \ldots \subseteq \mathfrak{A}_n \subseteq \ldots (n < \omega)$, is a chain of algebraic structures such that, for each $n < \omega$, $\mathfrak{A}_n$ is pure in $\mathfrak{A}_{n+1}$, then*

(i) *if there is an $n < \omega$ such that $\mathfrak{A}_n$ is weakly atomic compact, then $\bigcup_{n<\omega} \mathfrak{A}_n$ is also weakly atomic compact;*

(ii) *if there is a cofinal subsequence $\langle \mathfrak{A}_{i_n} \rangle_{n<\omega}$ of $\langle \mathfrak{A}_n \rangle_{n<\omega}$ such that $\mathfrak{A}_{i_n}$ is atomic compact, then $\bigcup_{n<\omega} \mathfrak{A}_n$ is a pure substructure of an atomic compact structure.*

Proof. (i) Let $\mathfrak{C}$ be a pure extension of $\bigcup_{n<\omega} \mathfrak{A}_n$. If the supposition of (i) holds, then $\mathfrak{C}$ is a pure extension of $\mathfrak{A}_n$. By Theorem 2.4 ((i) $\Leftrightarrow$ (ii)) [9], $\mathfrak{A}_n$ contains a homomorphic image of $\mathfrak{C}$. Hence $\bigcup_{n<\omega} \mathfrak{A}_n$ is weakly atomic compact.

(ii) Without loss of generality we can assume that $\mathfrak{A}_{i_n} = \mathfrak{A}_n$. Since each $\mathfrak{A}_n$ is a pure substructure of $\bigcup_{n<\omega} \mathfrak{A}_n$, thus there is in view of Theorem 2.3 ((ii) $\Leftrightarrow$ (iii)) of [9], a retraction h_n of $\bigcup_{n<\omega} \mathfrak{A}_n$ onto $\mathfrak{A}_n$ for all $n < \omega$. Moreover, let us observe that for each pair a_1, a_2 of distinct elements of $\bigcup_{n<\omega} \mathfrak{A}_n$ there is a structure $\mathfrak{A}_n$ such that $a_1, a_2 \in A_n$, and hence $h_n(a_1) \neq h_n(a_2)$. Thus there is a one-to-one homomorphism $h: \bigcup_{n<\omega} \mathfrak{A}_n \to \mathfrak{P}_{n<\omega} \mathfrak{A}_n$, defined by $h(a)(n) = h_n(a)$. It is visible that h is an isomorphism and that $h(\bigcup_{n<\omega} \mathfrak{A}_n)$ is a pure substructure of $\mathfrak{P}_{n<\omega} \mathfrak{A}_n$. Since all $\mathfrak{A}_n$ are atomic compact, hence such is also $\mathfrak{P}_{n<\omega} \mathfrak{A}_n$ and (ii) follows.

Now we give two examples showing that some refinements of Proposition 17 are not true.

EXAMPLE 18. *There is a chain of algebras $\mathfrak{A}_0 \subseteq \mathfrak{A}_1 \subseteq \ldots$, such that (i) each $\mathfrak{A}_n$ is atomic compact, and (ii) there is no algebra $\mathfrak{C} \supseteq \bigcup_{n<\omega} \mathfrak{A}_n$ that is weakly atomic compact.*

Indeed, let $\mathfrak{B} = \langle \omega, 0, 1, \cdot \rangle$, where 0 and 1 are constants and $\cdot$ is defined as follows: $x \cdot x = 0$ and $x \cdot y = 1$ if $x \neq y$. It is known (see [5] and [9]) that $\mathfrak{B}$ satisfies (ii). On the other hand, $\mathfrak{B}$ is the sum of an ascending chain of finite (and hence atomic compact) subalgebras, e.g. it suffices to put $\mathfrak{A}_n = \langle n+2, 0, 1, \cdot \rangle$.

EXAMPLE 19. *There is a chain of algebras $\mathfrak{A}_0 \subseteq \mathfrak{A}_1 \subseteq \ldots$, such that (i) each $\mathfrak{A}_n$ is atomic compact, (ii) for each $n < \omega$, $\mathfrak{A}_n$ is pure in $\mathfrak{A}_{n+1}$, and (iii) $\bigcup_{n<\omega} \mathfrak{A}_n$ is not atomic compact.*

Indeed, let $\mathfrak{B}$ be the free Boolean algebra with $\aleph_0$ free generators. It is visible that $\mathfrak{B}$ is not atomic compact (see e.g. [9], Theorem 4.1 ((i) $\Leftrightarrow$ (iv))). On the other hand, $\mathfrak{B}$ is the sum of an ascending chain of finite Boolean algebras. Finally, let us observe that if $\mathfrak{A} \subseteq \mathfrak{A}'$ are finite Boolean algebras, then $\mathfrak{A}$ is pure in $\mathfrak{A}'$, since it is a retract of $\mathfrak{A}'$.

Jan Mycielski constructed an example similar to 19 in which instead of (ii) the following stronger condition is fulfilled:

(ii') *for each $n < \omega$, $\mathfrak{A}_n \prec \mathfrak{A}_{n+1}$* [2].

But in his example all structures $\mathfrak{A}_n$ are of power $2^{\aleph_0}$. We do not know if the sum of a chain $\mathfrak{A}_0 \prec \mathfrak{A}_1 \prec \ldots \prec \mathfrak{A}_n \prec \ldots (n < \omega)$ of countable atomic compact structures must be atomic compact? (**P 627**)

EXAMPLE 20. *The algebra $\mathfrak{N} = \langle \omega, x+1 \rangle$ can not be represented as a union like in proposition 17 (ii), but $\mathfrak{N}$ is a pure substructure of $\langle \beta\omega, * \rangle$, where $\beta\omega$ is the Čech-Stone compactification of the set ω with the discrete topology and x^* is the (only) continuous extension of the function $x+1$ to $\beta\omega$.*

On the other hand, an algebraic structure having an atomic compact extension does not have in general an atomic compact extension in which it is pure. This is shown by the following example:

EXAMPLE 21. *The algebra $A = \langle \omega, f, g \rangle$, where f and g are any operations of one variable such that*

$$(f, g)\colon \omega \xrightarrow{\text{onto}} \omega \times \omega \setminus \{\langle k, k \rangle\colon k < \omega\}$$

has topologically (thus atomic) compact extensions (see e.g. [8], [9]) but it has no weakly atomic compact extensions in which $\mathfrak{A}$ is pure (i.e. downward pure, see (1)). A fortiori $\mathfrak{A}$ has no elementary extension which is weakly atomic compact (this solves in the negative a question formulated in [10]) [3].

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[2] $\prec$ denotes elementary extension, i.e. elementary purity.

[3] The idea of this example is due to C. Ryll-Nardzewski. Originally he has shown that this algebra $\mathfrak{A}$ cannot be a pure subalgebra of a topological compact algebra.

Indeed, let $\mathfrak{B}$ be an arbitrary weakly atomic compact extension of $\mathfrak{A}$. Let I be any set such that $|B| < |I|$ and consider the following set of equations:

$$\Lambda = \{''x_i = f(y_{ij})'' : i \neq j,\, i, j \in I\} \cup \{''x_j = g(y_{ij})'' : i \neq j,\, i, j \in I\}$$

(x_i and y_{ij} being unknowns, $i, j \in I$). It is visible that each finite subset of Λ has a solution in $\mathfrak{B}$ (since it has such a solution in $\mathfrak{A}$). Since $\mathfrak{B}$ is compact, Λ has a solution $\langle b_i \rangle_{i \in I}\ \langle c_{ij} \rangle_{i,j \in I}$ in $\mathfrak{B}$. Since $|B| < |I|$, there are $i, j \in I$, $i \neq j$, such that $b_i = b_j$. So $f(c_{ij}) = b_i = b_j = g(c_{ij})$. But this shows that the equation

$$(*) \qquad\qquad f(x) = g(x)$$

has a solution in $\mathfrak{B}$. On the other hand, by the definition of $\mathfrak{A}$, $(*)$ has no solution in $\mathfrak{A}$. Hence $\mathfrak{A}$ is not pure in $\mathfrak{B}$, q. e. d.

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