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A CONSTRUCTION IN BOOLEAN ALGEBRAS

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Introduction. An intrinsic description of the partial order in a dense subset S of non-zero elements of a Boolean algebra is due to Büchi [1]. Sikorski [3] gave another proof by constructing *a priori* the Stone space of the largest Boolean algebra which contains the partially ordered set S as a dense subset and is generated by S. A natural extension of this problem is to replace the requirement that S be dense, by the designation of a family $\mathscr{R}$ of subsets which have infima in the Boolean algebra, and a subfamily $\mathscr{N} \subseteq \mathscr{R}$ of subsets with infimum zero.

In this note a new method is introduced, namely the construction of a semigroup from the data and thence the space of semicharacters of the semigroup. It may be remarked that the existence of sufficiently many of these semicharacters does not require the axiom of choice, which intervenes only as Tychonoff's Theorem for the compactness of the topological space of semicharacters.

The symbols $\emptyset$ and 1 are used for the zero and unit elements, respectively, of each Boolean algebra; they are preserved under homomorphism. $\bigwedge$ and $\bigcap$ denote operations in abstract Boolean algebras and concrete fields of sets, respectively; the same convention holds for other pairs of symbols.

Denote by S a set, by $A, \ldots, E$ subsets of S, and by $\dot{s}$ the singleton containing the element s. Let $\leqslant$ be a partial order in S; the notation $A \leqslant B$, $A, B \subseteq S$, means that for each b in B there is some a in A for which $a \leqslant b$.

Let $\mathscr{R}$ be a family of non-void subsets of S that includes the singletons of S and is closed under the formation of unions. Let $\mathscr{N}$ be a subfamily of $\mathscr{R}$, possibly void, which contains no singletons, and if $A \leqslant B$ and $B \in \mathscr{N}$, then $A \in \mathscr{N}$.

A *Boolean mapping* of the system $(S, \leqslant, \mathscr{R}, \mathscr{N})$ is a function g of S into a Boolean algebra $\mathscr{F}$ such that

(i) If $s_1 \leqslant s_2$ in S, then $g(s_1) \leqslant g(s_2)$ in $\mathscr{F}$.

(ii) For each $A \in \mathscr{R}$, $\bigwedge g(A)$ exists in $\mathscr{F}$.

(iii) For each $A \in \mathscr{N}$, $\bigwedge g(A) = \emptyset$ in $\mathscr{F}$.

I. *There is a Boolean mapping g into an algebra $\mathscr{F}$ which is faithful in the sense that*

(i') $s_1 \leqslant s_2$ *if and only if* $g(s_1) \leqslant g(s_2)$.

(iii') $\bigwedge g(A) = \emptyset$ *if and only if* $A \in \mathscr{N}$.

Proof. For $A, B \in \mathscr{R}$, we write $A \simeq B$ if either $A \leqslant B \leqslant A$ in the order previously defined, or $A, B \in \mathscr{N}$. If A^* represents the $\simeq$-equivalence class of A, for each $A \in \mathscr{R}$, then the classes A^* and B^* determine uniquely the class $(A \cup B)^*$. Thus the collection $\mathscr{R}^*$ of classes forms a commutative semigroup under this composition, which we still denote as $\bigcup$. Let X be the set of all functions φ to $\{0,1\}$ defined on $\mathscr{R}^*$, which are multiplicative: $\varphi(A^* \cup B^*) \equiv \varphi(A^*)\cdot\varphi(B^*)$, and take the value zero on $\mathscr{N}^*$. X is provided with its product topology, in which it is a totally disconnected compact Hausdorff space. The elements of $\mathscr{R}^*$ are separated by X; a proof of this is given in Remark 1. We now take $\mathscr{F}$ as the Boolean algebra of X and define $g(s) = \{\varphi \in X \colon \varphi(\dot{s}^*) = 1\}$. The verification of (i'), (ii), (iii') follows.

(i') $g(s_1) \leqslant g(s_2)$ if and only if $\varphi(\dot{s}_1^*) = \varphi(\dot{s}_1^*)\varphi(\dot{s}_2^*)$ for every φ in X. This in turn means $\dot{s}_1^* = \dot{s}_1^* \cup \dot{s}_2^*$, or $\dot{s}_1 \leqslant \dot{s}_2$, which proves (i').

(ii) We prove that $\bigwedge g(A) = \{\varphi \in X \colon \varphi(A^*) = 1\}$. It is clear at least that the second set is a lower bound in $\mathscr{F}$ for the elements $g(s)$, $s \in A$. Let φ_0 be an interior point of $\bigcap g(A)$, so that for elements $A_1^*, A_2^*, \ldots, A_m^*$ of $\mathscr{R}^*$

$$W = \{\varphi \in X \colon \varphi(A_i^*) = \varphi_0(A_i^*),\ 1 \leqslant i \leqslant m\} \subseteq \bigcap_A g(s).$$

Since W does not contain the function identically 0, not every $\varphi_0(A_i^*)$ is 0. Moreover, those elements A_i^* for which $\varphi_0(A_i^*) = 1$ may be combined, so that

$$W = \{\varphi \in X \colon \varphi(A_1^*) = 1,\ \varphi(A_i^*) = 0,\ 2 \leqslant i \leqslant m\}.$$

Now let $\varphi \in W$, $\Psi \in X$, $\Psi(A_1^*) = 1$. Then the ordinary product $\varphi\cdot\Psi \in W$, hence $\varphi\cdot\Psi \in \bigcap_A g(s)$, whence also $\Psi \in \bigcap_A g(s)$. In different terms, $\Psi(A_1^*) = 1$ always implies $\Psi(\dot{s}^*) = 1$ for each s in A, yielding $A_1^* \cup \dot{s}^* = A_1^*$. Since $W \neq \emptyset$, $A_1 \notin \mathscr{N}$ and the equality $A_1^* \cup \dot{s}^* = A_1^*$ implies that $A_1 \leqslant \dot{s}$, $s \in A$. Then $A_1 \leqslant A$, whence

$$W \subseteq \{\varphi \in X \colon \varphi(A^*) = 1\}.$$

(iii') It is established that the interior of the set $\bigcap_A g(s)$ is equal to $\{\varphi \in X \colon \varphi(A^*) = 1\}$ and the latter subset is void if and only if $A^* \in \mathscr{N}^*$ or $A \in \mathscr{N}$.

Remark 1. The semigroup $\mathscr{R}^*$ is idempotent and commutative. For $u \in \mathscr{R}^*$ set $[u] = \{v \in \mathscr{R}^*: u \cup v = u\}$. Then the characteristic function of $[u]$ is multiplicative; $u \in [u']$ and $u' \in [u]$ if and only if $u = u'$. This is a simple case of [2], p. 78, of Hewitt and Zuckerman; cf. also Theorem 5.8 of [2]. In the present instance a non-constant semi-character [2] of $\mathscr{R}^*$ necessarily vanishes on $\mathscr{N}^*$, and is contained in X.

Remark 2. Evidently X can be identified with a family of functions θ on $\mathscr{R}$ to $\{0, 1\}$. The functions θ are precisely those which fulfill these requirements:

(m1) $\theta(A) \leqslant \theta(B)$ if $A \leqslant B$.

(m2) $\theta(A \cup B) = \theta(A) \cdot \theta(B)$, $A, B \in \mathscr{R}$.

(m3) $\theta = 0$ on $\mathscr{N}$.

II. *Let h be a Boolean mapping of $(S, \leqslant, \mathscr{R}, \mathscr{N})$ into an algebra $\mathscr{G}$, whose Stone space is Y. There is a unique Boolean homomorphism of $\mathscr{F}$ into $\mathscr{G}$ such that*

(a) $f \circ g = h$.

(b) $\bigwedge f \circ g(E) = f\big(\bigwedge g(E)\big)$ *for each* $E \in \mathscr{R}$.

Proof. It is equivalent but easier to construct the dual mapping $\hat{f}$ of Y into X. Condition (a) then becomes

(a′) $[\hat{f}(y)](\dot{s}^*) = 1$ if and only if $y \in h(s)$, where $y \in Y$, $s \in S$.

(b′) For each $E \in \mathscr{R}$ the set

$$\{y \in Y: [\hat{f}(y)](E^*) = 1\}$$

is exactly the interior of the set

$$\{y \in Y: [\hat{f}(y)](\dot{s}^*) = 1 \text{ for all } s \text{ in } E\}.$$

The equivalence of this with (b) depends on the previous calculations under (ii′).

Clearly (a′) and (b′) determine the mapping $\hat{f}$ uniquely, and the mapping $\hat{f}$ is necessarily continuous. Now for $y \in Y$ and $E \in \mathscr{R}$ define

$$\theta_y(E) = \begin{cases} 1, & \text{if} \quad y \in \bigwedge h(E), \\ 0, & \text{if} \quad y \notin \bigwedge h(E). \end{cases}$$

Then θ_y fulfills (m1)-(m3) of Remark 2, and may be construed as a function on $\mathscr{R}^*$. $\theta_y(\dot{s}^*) = 1$ if and only if $y \in h(s)$, which is (a′). Then $\{y \in Y: \theta_y(E^*) = 1\} = \bigcap_E h(s)$, and the interior of this intersection is $\bigwedge h(E)$, which is (b′).

Remark 3. It is not difficult to describe intrinsically the relation of $\mathscr{R}^*$ to $\mathscr{F}$. Suppose, indeed, that $\mathscr{F}$ is an algebra generated by a subset H which contains $\emptyset$ and 1, $H \wedge H = H$, and: whenever $h_1 \leqslant h_2 \vee \ldots \vee h_n$, $h_i \in H$, $1 \leqslant i \leqslant n$, then $h_1 \leqslant h_i$ for some $i > 1$. If H is the union of two disjoint subsets H_1 and H_2, with $\emptyset \in H_1$, $1 \in H_2$, $H \wedge H_1 = H_1$, $H_2 \wedge H_2 = H_2$, then there is a prime ideal J of $\mathscr{F}$ such that $J \cap H = H_1$, namely the smallest ideal of $\mathscr{F}$ which contains H_1. Thus the Stone space X of $\mathscr{F}$ is the space of semicharacters of the semigroup H which map 1 to 1 and $\emptyset$ to 0.

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