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SOME REMARKS ON POST ALGEBRAS

BY

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Introduction. The theory of Post algebras has been enormously simplified in recent years notably by Epstein [5], Balbes and Dwinger [1] and [2], Rousseau [11], Speed [12], and Traczyk [14] and [15]. In this paper we take, as a starting point, the isomorphism of the categories of (pseudo-) Post algebras and (pseudo-) Boolean algebras; drawing from this the isomorphism of the various structure lattices of a (pseudo-) Post algebra with those of the underlying (pseudo-) Boolean algebra. As an application we characterize the congruence lattice of a (pseudo-) Post algebra P and show that it is Boolean if and only if P is a finite Post algebra. In Section 3 we show that the MacNeille completion of a (pseudo-) Post algebra P is a (pseudo-) Post algebra, the underlying (pseudo-) Boolean algebra of which is the completion of the underlying (pseudo-) Boolean algebra of P. In addition, we characterize the essential extensions in the category of Post algebras. The objective of Section 4 is the characterization of Post functions and generalization of results due to Grätzer [8] on Boolean functions. Finally, we define and characterize profinite Post algebras.

1. Preliminary definitions. A *pseudo-Boolean algebra* D (Heyting algebra) is a lattice with the least element 0, equipped with a binary operation $*$, called *relative pseudo-complementation*, satisfying

$$x \wedge a \leqslant b \text{ if and only if } x \leqslant a * b.$$

Such algebras are known to be equationally definable, distributive, have least and greatest elements 0, 1 and pseudo-complemented; the element $a^* = a*0$ being the pseudo-complement of a in D. An element $a \in D$ satisfying $a^* = 0$ is said to be *dense*. The set $D^{**} = \{x \in D;\ x = x^{**}\}$ is well known to be a Boolean algebra, coinciding with D if and only if 1 is the only dense element in D.

A *(pseudo-) Post algebra of order n* can be defined as the free product (coproduct) of a (pseudo-) Boolean algebra and a finite chain E of length n in the category $\mathscr{D}_{0,1}$ of distributive lattices with 0, 1.

An equational definition, due to Rousseau [11], is as follows:

An algebra

$$P = \langle P; \wedge, \vee, *; {}^*, D_1, \ldots, D_{n-1}; e_0, \ldots, e_{n-1}\rangle$$

with three binary ($\wedge, \vee, *$), n unary ($^*, D_1, \ldots, D_{n-1}$) and n nullary ($e_0, \ldots, e_{n-1}$) operations is a pseudo-Post algebra of order n if it satisfies ($i = 1, \ldots, n-1$)

[P.1] $\langle P; \wedge, \vee, *; {}^*\rangle$ is a pseudo-Boolean algebra,

[P.2] $D_i(x \wedge y) = D_i(x) \wedge D_i(y)$,

[P.3] $D_i(x \vee y) = D_i(x) \vee D_i(y)$,

[P.4] $D_i\big(D_k(x)\big) = D_k(x)$ ($k = 1, 2, \ldots, n-1$),

[P.5] $D_i(e_k) = \begin{cases} 1, & i \leqslant k, \\ 0, & i > k, \end{cases}$

[P.6] $x = \bigvee_{i=1}^{n-1} D_i(x) \wedge e_i$.

A *Post algebra of order n* is a pseudo-Post algebra satisfying the additional identity

[P.7] $D_1(x) \vee D_1^*(x) = 1$.

In any (pseudo-) Post algebra P the following are true:

(i) The set $E = \{e_0, e_1, \ldots, e_{n-1}\}$ forms a chain $0 = e_0 < e_1 < \ldots < e_{n-1} = 1$ (providing $|P| > 2$) and is a pseudo-Boolean subalgebra of P.

(ii) The mappings D_i have common image D, which is a (pseudo-) Boolean subalgebra of P, and reduce to the identity on D.

(iii) $D_i(x * y) = \bigwedge_{j=1}^{i} \big(D_j(x) * D_j(y)\big)$.

(iv) $D_i(x) \leqslant D_j(x)$ whenever $j \leqslant i$ ($i, j = 1, 2, \ldots, n-1$).

(v) $x \leqslant y$ if and only if $D_i(x) \leqslant D_i(y)$ for all $i = 1, 2, \ldots, n-1$.

(vi) The elements $D_i(x)$ are unique in that they are the only elements in D satisfying [P.6] and (iv).

(vii) P is the free product of D and E in $\mathscr{D}_{0,1}$.

If a (pseudo-) Post algebra of order n has underlying (pseudo-) Boolean algebra D, we denote it by $[D]_n$.

If $\mathscr{P}_n$ denotes the category of (pseudo-) Post algebras (of order n) and homomorphisms, $\mathscr{B}$ denotes the category of (pseudo-) Boolean algebras and homomorphisms, then the functors $F: \mathscr{B} \to \mathscr{P}_n$ and $G: \mathscr{P}_n \to \mathscr{B}$ given by $F(D) = [D]_n$ and $F(h) = [h]_n$, where (cf. [11], Theorem 7)

$$[h]_n(x) = \bigvee_{i=1}^{n-1} h\big(D_i(x)\big) \wedge e_i,$$

$G(P)$ is the underlying (pseudo-) Boolean algebra D of P and $G(h)$ is the restriction of h to D, are mutually inverse. Hence $\mathscr{P}_n$ and $\mathscr{B}$ are isomorphic categories.

A *(pseudo-) Post filter* in a (pseudo-) Post algebra P is a lattice filter ∇ closed under the mapping D_{n-1}. The set $\mathscr{F}_P$ of all (pseudo-) Post filters is a complete sublattice of the lattice $\langle \mathscr{F}; \bigcap, \coprod \rangle$ of all lattice filters in P. A *filter of order* i is a lattice filter containing e_i but not e_{i-1}.

A *(pseudo-) Post congruence* is a pseudo-Boolean congruence Φ satisfying the additional property

$$a \equiv b(\Phi) \text{ implies } D_{n-1}(a) \equiv D_{n-1}(b)(\Phi).$$

The set $\mathscr{K}_P$ of all (pseudo-) Post congruences is a complete sublattice of the lattice $\langle \mathscr{K}; \bigcap, \coprod \rangle$ of all pseudo-Boolean congruences.

Clearly, $\Phi \in \mathscr{K}_P$ implies that $\operatorname{Ker}\Phi = \{x \in P; x \equiv 1(\Phi)\} \in \mathscr{F}_P$.

2. Congruences, filters and subalgebras. The result of this section contains a clarification and simplification of early results in the field.

THEOREM 2.1. *In any (pseudo-) Post algebra $P = [D]_n$ the lattices $\mathscr{K}_P$, $\mathscr{F}_P$, $\mathscr{K}_D$, $\mathscr{F}_D$ are mutually isomorphic pseudo-Boolean algebras. The lattice $\mathscr{S}_P$ of (pseudo-) Post subalgebras of P and the lattice $\mathscr{S}_D$ of (pseudo-) Boolean subalgebras of D are isomorphic.*

Proof. The isomorphism of $\mathscr{K}_D$ and $\mathscr{F}_D$ is well known as is the fact that if $\nabla \in \mathscr{F}_P$, then the relation $\Theta[\nabla]$ defined by

$$x \equiv y(\Theta[\nabla]) \text{ if and only if } x \circ y = (x * y) \wedge (y * x) \in \nabla$$

is a pseudo-Boolean congruence on P (cf. [10]). In addition, $x \circ y \in \nabla$ implies that

$$D_{n-1}(x * y) \wedge D_{n-1}(y * x) = D_{n-1}(x \circ y) \in \nabla,$$

or, equivalently,

$$\bigwedge_{j=1}^{n-1} \big(D_j(x) \circ D_j(y)\big) = \bigwedge_{j=1}^{n-1} \big(D_j(x) * D_j(y)\big) \wedge \bigwedge_{j=1}^{n-1} \big(D_j(y) * D_j(x)\big) \in \nabla,$$

and, therefore, $D_j(x) \circ D_j(y) \in \nabla$ for all $j = 1, \ldots, n-1$. It follows that $\Theta[\nabla] \in \mathscr{K}_P$. That the mapping $\nabla \to \Theta[\nabla]$ is an order isomorphism from $\mathscr{F}_P$ onto $\mathscr{K}_P$ follows from the fact that $\nabla = \operatorname{Ker}\Theta[\nabla]$ and $\Phi \in \mathscr{K}_P$ implies $\Phi = \Theta[\operatorname{Ker} \varphi]$.

The isomorphism of $\mathscr{F}_P$ and $\mathscr{F}_D$ follows from the fact that if $\nabla \in \mathscr{F}_D$ and

$$[\nabla]_n = \{x \in P; D_{n-1}(x) \in \nabla\},$$

then the mapping $\nabla \to [\nabla]_n$ is an order isomorphism. The details are left to the reader.

Finally, if $S \in \mathscr{S}_D$, then the sublattice $[S \cup E]$ of P, generated by S and E, is the free product $[S]_n$ of S and E in $\mathscr{D}_{0,1}$ (cf. [7], Theorem 12.5) and, therefore, a (pseudo-) Post subalgebra of P. Clearly, the mapping $S \to [S]_n$ from $\mathscr{S}_D$ into $\mathscr{S}_P$ is order-preserving; that it is onto follows from the observation that if $T \in \mathscr{S}_P$, then $S = T \cap D \in \mathscr{S}_D$ and $T = [S]_n$. Hence $\mathscr{S}_P$ and $\mathscr{S}_D$ are isomorphic.

COROLLARY 2.2. *In any Post algebra P the following are equivalent:*

(i) *P is simple,*

(ii) *P is subdirectly irreducible,*

(iii) $P \cong [2]_n$.

COROLLARY 2.3. *If P is a (pseudo-) Post algebra, then*

(i) *$V \in \mathscr{F}_P$ is maximal if and only if, for every $x \in P$, exactly one of $D_{n-1}(x)$, $D^*_{n-1}(x)$ is in V* (cf. [10], Chapter I, 13.10);

(ii) *a prime filter is a (pseudo-) Post filter if and only if it is of order $n-1$.*

COROLLARY 2.4. *If P is a Post algebra, then*

(i) *$V \in \mathscr{F}_P$ is maximal if and only if it is prime;*

(ii) *a prime filter V is a Post filter if and only if it is minimal.*

COROLLARY 2.5. *If Q is a subalgebra of a (pseudo-) Post algebra P, then any (pseudo-) Post congruence on Q has an extension to P.*

Proof. It suffices to show that the category of (pseudo-) Boolean algebras has the congruence extension property. Let D be a (pseudo-) Boolean algebra, $S \in \mathscr{S}_D$ and $\Phi \in \mathscr{K}_S$. If

$$V_S = \operatorname{Ker}\Phi \quad \text{and} \quad V = \{x \in D;\ x \geqslant d \text{ for some } d \in V_S\},$$

then, clearly, $V \in \mathscr{F}_D$, $V_S = V \cap S$ and $\Phi = \Theta[V] \cap S^2$, so that $\Theta[V]$ is the required extension.

For Post algebras, Corollary 2.5 follows from the fact that every Post algebra can be embedded in an injective Post algebra (cf. [1]) and that every equational class of algebras satisfying this condition has the congruence extension property (cf. [7]). Incidentally, since $\mathscr{P}_n$ and $\mathscr{B}$ are isomorphic categories, a pseudo-Post algebra $P = [D]_n$ is injective if and only if D is injective in $\mathscr{B}$. The injectives in $\mathscr{B}$ are the complete Boolean algebras (cf. [3]) and, therefore, P is injective if and only if it is a complete Post algebra.

The next theorem contains a characterization of the congruence lattice of a pseudo-Post algebra.

THEOREM 2.6. *A lattice L is the lattice of pseudo-Post filters in a pseudo-Post algebra if and only if it is an algebraic lattice in which the compact elements form a sublattice C whose dual $\breve{C}$ is pseudo-Boolean.*

Proof. It suffices to prove the theorem for pseudo-Boolean algebras. The necessity follows from the fact that the compact elements in the lattice $\mathscr{F}_D$ of filters of a pseudo-Boolean algebra D are precisely the principal filters which form a lattice dually isomorphic to D. The sufficiency is proved by showing that if L satisfies the conditions of the theorem, then $L \cong \mathscr{F}_{\breve{C}}$ under the mapping φ: $\nabla \to \bigvee\{f; f \in \nabla\}$ from $\mathscr{F}_{\breve{C}}$ to L. Clearly, φ preserves order and, since $a = \bigvee\{x; x \in \nabla_a\}$, where ∇_a is the filter $\{c \in C; c \leqslant a\}$ in $\breve{C}$, it follows that φ is onto L. Finally, if $\nabla_1\varphi \subseteq \nabla_2\varphi$ and $c_1 \in \nabla_1$, then $c_1 \in C$ and $c_1 \leqslant \bigvee\{f; f \in \nabla_2\}$, so that $c \leqslant \bigvee\{f; f \in F_2\}$ for some finite subset F_2 of ∇_2, and, therefore, $c_1 \in \nabla_2$. Hence φ is an isomorphism.

THEOREM 2.7. *If $P = [D]_n$ is a (pseudo-) Post algebra, then $\mathscr{F}_P$ is Boolean if and only if P is a finite Post algebra.*

Proof. Clearly, it suffices to show that $\mathscr{F}_D$ is Boolean if and only if D is a finite Boolean algebra. If $\mathscr{F}_D$ is Boolean, then the pseudo-complement

$$\nabla^* = \{x \in D; x \vee a = 1 \text{ for all } a \in \nabla\}$$

of ∇ in $\mathscr{F}_D$ coincides with the complement of ∇ in $\mathscr{F}_D$ and, therefore, $\nabla \amalg \nabla^* = D$, so that $a \wedge b = 0$ for some $a \in \nabla, b \in \nabla^*$. It follows that b has a as a complement and, therefore, $a = b^*$ so that $a \in D^{**}$. Hence, since $b = a^*$, we have shown that there exists an $a \in D^{**}$ with the property that $a \in \nabla$ and $a^* \in \nabla^*$. Consequently, $a^* \vee p = 1$ for all $p \in \nabla$, and, therefore, $a = a \wedge (a^* \vee p) = a \wedge p$, so that $a \leqslant p$ for all $p \in \nabla$. In summary, every filter in D is principal and generated by an element in D^{**}. Therefore, $D = D^{**}$ so that D is a Boolean algebra in which every filter is principal, which implies that D is finite.

3. The MacNeille completion. If X is a non-empty subset of a poset P, we write X^+ (X^-) for the set of all upper (lower) bounds of X in P and call X *normal* if $X = X^{+-}$. The *MacNeille completion* of P is the complete lattice $\mathscr{N}(P)$ of all normal subsets of P in which the greatest lower bound and least upper bound of $\{N_a; a \in A\} \subseteq \mathscr{N}(P)$ are

$$\bigcap_{a \in A} N_a \text{ and } \nabla_{a \in A} N_a = \Big(\bigcup_{a \in A} N_a\Big)^{+-},$$

respectively.

We write $(a)\downarrow$ for $\{a\}^-$ and call sets of the form $\bigcap_{a \in A} (a_a)\downarrow$ *comprincipal ideals.*

Epstein [5] has shown, by topological methods, that the class of Post algebras is closed under the process of forming MacNeille completions. Recently, Balbes and Dwinger [1] gave a categorical characterization of the MacNeille completion of a Post algebra.

LEMMA 3.1. *A subset N of a poset is normal if and only if it is a comprincipal ideal. More specifically, if N is normal, then*

$$N = \bigcap \{(a)\downarrow; a \in N^+\}.$$

It is well known (cf. [4]) that the set of all ideals in a distributive lattice L with 0 forms a complete pseudo-Boolean algebra $\mathscr{I}$; the relative pseudo-complement $\Delta_1 * \Delta_2$ of Δ_1, Δ_2 in $\mathscr{I}$ being $\{x \in L;\ a \wedge x \in \Delta_2$ for all $a \in \Delta_1\}$.

LEMMA 3.2. *The set of all comprincipal ideals in a pseudo-Boolean algebra is closed under relative pseudo-complementation.*

Proof. In fact, if $\Delta_2 = \bigcap_{a \in A} (a_a)\downarrow$ and Δ_1 is any ideal, then

$$\Delta_1 * \Delta_2 = \bigcap \{(a * a_a)\downarrow;\ a \in A,\ a \in \Delta_1\}.$$

For, if $x \in \Delta_1 * \Delta_2$, then, for all $a \in \Delta_1$, there exists a $b \in \Delta_2$ such that $a \wedge x \leqslant b$ or, equivalently, $x \leqslant a * b$, and so $x \leqslant a * a_a$ for all $a \in A$, $a \in \Delta_1$. If, conversely, $x \leqslant a * a_a$ or, equivalently, $a \wedge x \leqslant a_a$ for all $a \in A$, $a \in \Delta_1$, then $a \wedge x \in \Delta_2$ for all $a \in \Delta_1$, so that $x \in \Delta_1 * \Delta_2$.

THEOREM 3.3. *The MacNeille completion of a pseudo-Post algebra P is a pseudo-Post algebra.*

Proof. We start by showing that

$$\bigcap_{a \in A} (a_a)\downarrow = \nabla_{k=1}^{n-1} \Big\{ \bigcap_{a \in A} (D_k(a_a))\downarrow \cap (e_k)\downarrow \Big\}.$$

Clearly, it suffices to show that if

$$S = \bigcup_{k=1}^{n-1} \Big\{ \bigcap_{a \in A} (e_k \wedge D_k(a_a))\downarrow \Big\},$$

then

$$S^{+-} = \bigcap_{a \in A} (a_a)\downarrow.$$

If $x \in S$, then there exists a k such that $x \leqslant e_k \wedge D_k(a_a)$ for all $a \in A$ and so, since $e_k \wedge D_k(a_a) \leqslant a_a$, it follows that $a_a \in S^+$ for all $a \in A$ and, therefore, $S^{+-} \subseteq \bigcap_{a \in A} (a_a)\downarrow$. For the reverse inclusion, suppose $l \leqslant a_a$ for all $a \in A$ and let $u \in S^+$. Clearly, $u \in S_k^+$, where

$$S_k = \bigcap_{a \in A} (e_k \wedge D_k(a_a))\downarrow$$

and consists of those $x \in P$ for which $D_i(x) = 0$ for $i > k$ and $D_i(x) \leqslant D_k(a_a)$ for all $i \leqslant k$, $a \in A$. Therefore,

$$x_k = \bigvee_{j=1}^{k} (e_j \wedge D_j(l)) \in S_k,$$

so that $u \geqslant x_k$ which implies that

$$u \geqslant \bigvee_{k=1}^{n-1} x_k = \bigvee_{i=1}^{n-1} (e_i \wedge D_i(l)) = l$$

or, equivalently, $l \in S^{+-}$.

It remains only to show that if $N \in \mathcal{N}(P)$ and operations D_k on $\mathcal{N}(P)$ into itself are defined by

$$D_k(N) = \bigcap\{(D_k(a))\downarrow;\ a \in N^+\},$$

then [P.2]-[P.5] are satisfied. We prove only [P.3] leaving the easier proofs to the reader.

Let M denote the set $D_k(N_1) \cup D_k(N_2)$ and N the normal set $D_k(N_1) \nabla D_k(N_2)$, so that $N^+ = M^{+-+} = M^+$. We first show that $N \subseteq D_k(N_1 \nabla N_2)$. Suppose $x \in N$, so that $x \leqslant u$ for all $u \in N^+$; then we must prove that

$$x \leqslant D_k(a) \quad \text{for all } a \in (N_1 \nabla N_2)^+ = (N_1 \cup N_2)^+,$$

and for this it suffices to show that $D_k(a) \in M^+$ for all $a \in (N_1 \cup N_2)^+$. Now, if $z \in M$, so that

$$z \in (D_k(a_1))\downarrow \cup (D_k(a_2))\downarrow \quad \text{for all } a_1 \in N_1^+,\ a_2 \in N_2^+,$$

then (since $(N_1 \cup N_2)^+ \subseteq N_1^+, N_2^+$) $z \in (D_k(a))\downarrow$ and, therefore, $D_k(a) \in M^+$ for all $a \in (N_1 \cup N_2)^+$. The reverse inclusion follows on observing that

$$(N_1 \nabla N_2)^+ \subseteq N_1^+, N_2^+ \text{ implies } D_k(N_1), D_k(N_2) \subseteq D_k(N_1 \nabla N_2).$$

COROLLARY 3.4. $\mathcal{N}([D]_n) \cong [\mathcal{N}(D)]_n$.

An extension E of an algebra A is said to be *essential* if and only if, for every algebra B, any homomorphism $\varphi\colon E \to B$ whose restriction to A is one-to-one is itself one-to-one.

An extension E of a lattice L is said to be *meet-dense* if and only if every element in E is a meet of elements in L.

In [1] essential injective extensions in $\mathcal{P}_n$ are characterized; they are the MacNeille completions. A characterization of essential extensions in $\mathcal{P}_n$ is given in the following

THEOREM 3.5. *An extension of a Post algebra is essential if and only if it is meet-dense.*

Proof. From the isomorphism of the categories $\mathcal{P}_n$ and $\mathcal{B}$ it follows that if $P_1, P_2 \in \mathcal{P}_n$, then P_1 is an essential extension of P_2 if and only if $G(P_1)$ is an essential extension of $G(P_2)$. That P_1 is a meet-dense extension of P_2 if and only if $G(P_1)$ is a meet-dense extension of $G(P_2)$ follows easily from Theorems 2.2 and 2.4 of [5]. Consequently, it suffices to prove the theorem in the category $\mathcal{B}$. Let $B_1, B_2 \in \mathcal{B}$ and suppose B_1 is a meet-dense extension of B_2. We show that $V = \{1\}$ is the only filter in B_1 satisfying $V \cap B_2 = \{1\}$. If V satisfies the condition and $x \in V$, then, since $x = \bigwedge\{d;\ d \in B\}$ for some subset B of B_2 and V is a filter, it follows that $B \subseteq V \cap B_2 = \{1\}$ and, therefore, $x = 1$, so that $V = \{1\}$. Therefore, B_1 is an essential extension of B_2.

Conversely, if B_1 is an essential extension of B_2, then, since $\mathscr{N}(B_2)$ is injective in $\mathscr{B}$, it can be embedded over B_2 into $\mathscr{N}(B_2)$. It follows, since $\mathscr{N}(B_2)$ is a meet-dense extension of B_2, that B_1 is a meet-dense extension of B_2.

4. Post functions and equations. A *Post function of m variables* on a Post algebra $P = [D]_n$ is one obtained from the constant functions $a(x_1, \ldots, x_m) = a$ and the identity functions $f_i(x_1, \ldots, x_m) = x_i$ by a finite number of applications of the operations $\wedge$, $\vee$, *, $D_1, \ldots, D_{n-1}$.

A function $f(x_1, \ldots, x_m)$ on a Post algebra P is said to be *congruence-preserving* if, for all $\Phi \in \mathscr{K}_P$,

$$a_i \equiv b_i(\Phi) \text{ implies } f(a_1, \ldots, a_m) \equiv f(b_1, \ldots, b_m)(\Phi)$$

and *isotonic on* $S \subseteq P$ if

$$a_i \leqslant b_i \ (a_i, b_i \in S) \text{ implies } f(a_1, \ldots, a_m) \leqslant f(b_1, \ldots, b_m).$$

Grätzer [8] showed that, for Boolean algebras, the property of being congruence-preserving characterizes the Boolean functions. We generalize this to Post algebras. Descriptions of congruence-preserving functions on distributive lattices and p-rings may be found in [6] and [9], respectively.

THEOREM 4.1. *If $f(x_1, \ldots, x_m)$ and $g(x_1, \ldots, x_m)$ are congruence-preserving functions on a Post algebra $[D]_n$ into itself, then $f = g$ identically if and only if f and g agree on E.*

Proof. The necessity of the condition is obvious. To prove the sufficiency suppose f, g agree on E but $p = f(a_1, \ldots, a_m) \neq g(a_1, \ldots, a_m) = q$; then there exists a $k \in \boldsymbol{n}$ such that $p_k = D_k(p) \neq D_k(q) = q_k$, and so there exists a Boolean homomorphism h from D onto 2 distinguishing p_k and q_k. The extension $[h]_n \colon [D]_n \to [2]_n$ is onto and distinguishes p and q. Now, since f is congruence-preserving, we can define a function f_n on $[2]_n$ to itself by

$$f_n\big([h]_n(x_1), \ldots, [h]_n(x_m)\big) = [h]_n\big(f(x_1, \ldots, x_m)\big).$$

Therefore, since $[h]_n(e_0), \ldots, [h]_n(e_{n-1})$ are the only elements in $[2]_n$ and f, g agree on E, we have $f_n = g_n$ identically, which contradicts $[h]_n(p) \neq [h]_n(q)$.

A simple application of Theorem 4.1 yields the following

COROLLARY 4.2. *The class $\mathscr{P}_n$ of Post algebras of order n is equationally complete.*

Proof. If $\boldsymbol{p}, \boldsymbol{q}$ are polynomial symbols in $\mathscr{P}_n$ and $\boldsymbol{q} = \boldsymbol{p}$ holds in some algebra $P \in \mathscr{P}_n$, then $\boldsymbol{p} = \boldsymbol{q}$ is an identity in the chain of constants of P and, therefore, an identity in the class $\mathscr{P}_n$.

THEOREM 4.3. *A function on a Post algebra P into itself is a Post function if and only if it is congruence-preserving.*

Proof. The necessity is obvious. To prove the sufficiency let $f(x_1, \ldots, x_m)$ be a congruence-preserving function on P into itself and let $g(x_1, \ldots, x_m)$ be the Post function

$$\bigvee_{n^m} \{f(e_{i_1}, \ldots, e_{i_m}) \wedge \bigwedge_{k=1}^{m} C_{i_k}(x_k)\},$$

where $C_0(x) = D_1^*(x)$, $C_i(x) = D_i(x)D_{i+1}^*(x)$ $(i = 1, 2, \ldots, n-1)$ and the join $\bigvee_{n^m}$ is taken over all ordered m-tuplets $\langle i_1, \ldots, i_m\rangle \in \boldsymbol{n}^m$. For each tuplet $\langle j_1, \ldots, j_m\rangle \in \boldsymbol{n}^m$,

$$g(e_{j_1}, \ldots, e_{j_m}) = \bigvee_{n^m} \{f(e_{i_1}, \ldots, e_{i_m}) \wedge \bigwedge_{k=1}^{m} C_{i_k}(e_{j_k})\},$$

and so, since

$$\bigwedge_{k=1}^{m} C_{i_k}(e_{j_k}) \neq 0$$

only if $i_k = j_k$ $(k = 1, \ldots, m)$ when it equals 1, it follows that f and g agree on E and, therefore, $f = g$ identically.

It is well known that a function on a Boolean algebra into itself is isotonic if and only if it is a lattice function.

A *D-function* on a Post algebra $[D]_n$ is one which can be obtained from the constant functions and identity functions by a finite number of applications of the operations $\wedge, \vee, D_1, \ldots, D_{n-1}$.

THEOREM 4.4. *A Post function is isotonic if and only if it is a D-function.*

Proof. The sufficiency is obvious. To prove the necessity let $f(x_1, \ldots, x_m)$ be an isotonic Post function and let $g(x_1, \ldots, x_m)$ be the D-function

$$\bigvee_{n^m} \{f(e_{i_1}, \ldots, e_{i_m}) \wedge \bigwedge_{k=1}^{m} D_{i_k}(x_k)\}.$$

For each tuplet $\langle j_1, \ldots, j_m\rangle \in \boldsymbol{n}^m$,

$$g(e_{j_1}, \ldots, e_{j_m}) = \bigvee_{n^m} \{f(e_{i_1}, \ldots, e_{i_m}) \wedge \bigwedge_{k=1}^{m} D_{i_k}(e_{j_k})\}$$

in which the only non-zero terms are those where $i_k \leqslant j_k$ $(k = 1, \ldots, m)$ and for these $D_{i_k}(e_{j_k}) = 1$ and $f(e_{i_1}, \ldots, e_{i_m}) \leqslant f(e_{j_1}, \ldots, e_{j_m})$. Hence f and g agree on E and, therefore, $f = g$ identically.

COROLLARY 4.5. *A Post function is isotonic if and only if it is isotonic on E.*

5. Profinite Post algebras. If $P = [D]_n$ is a Post algebra, then the following are known to be equivalent (cf. [1] and [5]):

(i) $P \cong \boldsymbol{n}^S$ for some set S.

(ii) P is complete and atomic.

(iii) D is complete and atomic.

We give an alternative description of such algebras in terms of inverse limits.

If $\{P_a; \varphi_{a\beta} \mid a \in A\}$ is an inverse system of algebras P_a and homomorphisms $\varphi_{a\beta}: P_a \to P_\beta$ $(a, \beta \in A, \beta \leqslant a)$, where A is a directed set and $\varphi: P \to \underleftarrow{\mathrm{Lim}}\, P_a$ is an isomorphism, then the homomorphisms

$$\varphi_a = p_a \circ \varphi: P \to P_a,$$

where p_a: $\underleftarrow{\mathrm{Lim}}\, P_a \to P_a$ denotes the a-th projection, will be called *decomposition homomorphisms*. Clearly, given such a representation of P, we can obtain one in which each φ_a and $\varphi_{a\beta}$ is surjective. An algebra having an inverse limit representation in which no decomposition homomorphism is an isomorphism will be called *inversely reducible*. It is easy to see that every finite algebra is inversely irreducible.

We call an algebra *profinite* if it can be represented as an inverse limit of finite algebras in which no decomposition homomorphism is an isomorphism.

LEMMA 5.1. *If* $\{[D_a]_n; \varphi_{a\beta} \mid a \in A\}$ *is an inverse system of (pseudo-) Post algebras, then* $\{D_a; G(\varphi_{a\beta}) \mid a \in A\}$ *is an inverse system of (pseudo-) Boolean algebras and* $\underleftarrow{\mathrm{Lim}}\, [D_a]_n = [\underleftarrow{\mathrm{Lim}}\, D_a]_n$.

The proof follows from the observation that $\underleftarrow{\mathrm{Lim}}\, D_a$ is the image of $\underleftarrow{\mathrm{Lim}}\, [D_a]_n$ under the mapping D_{n-1}.

THEOREM 5.2. *An infinite Post algebra is profinite if and only if it is complete and atomic.*

Proof. It suffices, by the lemma, to prove the statement of the theorem for Boolean algebras. To prove sufficiency, let B be a complete and atomic Boolean algebra, $\mathscr{A}_B$ its set of atoms and $\mathscr{F} = \{f_a;\ a \in A\}$ the set of finite joins of members of $\mathscr{A}_B$. Then $\mathscr{F}$ is a sublattice of B with

$$\bigvee_{a \in A} f_a = 1$$

and, since B is infinite, $\mathscr{F}$ cannot contain 1. If we partially order the index set A by $\beta \leqslant a$ if and only if $f_\beta \leqslant f_a$, then A is a directed set indexing the non-trivial congruences $\Phi_a = \Theta[\{f_a\}^+]$ in such a way that $\Phi_a \subseteq \Phi_\beta$ whenever $\beta \leqslant a$. Consequently, the system of quotient algebras $A_a = B/\Phi_a$ and homomorphisms $\varphi_{a\beta}: A_a \to A_\beta$ defined by $\varphi_{a\beta}([x]_a) = [x]_\beta$ whenever $\beta \leqslant a$ (where $[x]_a$ denotes the congruence class $\mathrm{mod}\, \Phi_a$ containing x) form an inverse system. We show that each A_a is finite. The congruence lattice $\mathscr{K}_a$ of A_a is isomorphic to the sublattice $\{\Phi_a\}^+$ of $\mathscr{K}_B$ under the mapping $\Phi \to \Phi/\Phi_a$ $(\Phi \in \{\Phi_a\}^+)$, where $\Phi/\Phi_a \in \mathscr{K}_a$ is defined by $[a]_a \equiv [b]_\beta(\Phi/\Phi_a)$ if and only if $a \equiv b(\Phi)$. Therefore, the subdirect product representations of A_a correspond one-to-one with the sets of elements in the lattice $\{\Phi_a\}^+$ whose meet is Φ_a. If

$$f_a = \bigvee_{i=1}^{m} a_{a_i}, \quad \text{where } a_{a_i} \in \mathscr{A}_B,$$

then such a set is $\theta_{a_i} = \Theta[\{a_{a_i}\}^+]$ $(i = 1, 2, \ldots, m)$ and, therefore, A_a is a subdirect product of $A_a/(\theta_{a_i}/\Phi_a) \cong B/\theta_{a_i}$ $(i = 1, 2, \ldots, m)$ which shows, since θ_{a_i} is maximal, that A_a is finite. Clearly,

$$\bigcap_{a \in A} \Phi_a = \omega_0$$

(the trivial congruence) and, therefore, the correspondence $\varphi: B \to \overleftarrow{\mathrm{Lim}}\, A_a$, defined by $[\varphi(x)](a) = [x]_a$ for all $a \in A$, is an embedding of B in $\overleftarrow{\mathrm{Lim}}\, A_a$.

It remains only to show that φ is surjective. Let $t \in \overleftarrow{\mathrm{Lim}}\, A_a$ and write $t(a) = [x_a]_a$; then it suffices to show that there exists an $x \in B$ such that $[x]_a = [x_a]_a$ for all $a \in A$. Now, $x \circ x_a \geqslant f_a$ is, by a direct calculation, equivalent to

$$f_a \wedge x_a \leqslant x \leqslant f_a * x_a$$

and, therefore, φ is surjective if and only if

$$\bigvee_{a \in A} (f_a \wedge x_a) \leqslant \bigwedge_{a \in A} (f_a * x_a).$$

An equivalent condition is that

$$f_a \wedge x_a \leqslant f_\beta * x_\beta \quad \text{for all } a, \beta \in A$$

or, again,

$$f_a \wedge f_\beta \leqslant x_a * x_\beta \quad \text{for all } a, \beta \in A.$$

On interchanging the roles of a, β, we have the equivalent form

$$f_a \wedge f_\beta \leqslant x_a \circ x_\beta \quad \text{for all } a, \beta \in A.$$

However, $f_a \wedge f_\beta \in \mathscr{F}$, so that $f_a \wedge f_\beta = f_\gamma$ for some $\gamma \in A$, namely $\gamma = a \wedge \beta$. Therefore, φ is surjective if and only if $x_a \equiv x_\beta(\Phi_{a \wedge \beta})$ or, equivalently, $[x_a]_{a \wedge \beta} = [x_\beta]_{a \wedge \beta}$ for all $a, \beta \in A$. Finally, $\gamma \leqslant a, \beta$ implies that

$$[x_\gamma]_\gamma = t(\gamma) = \varphi_{\gamma a}\big(t(a)\big) = \varphi_{\gamma a}([x_a]_a) = [x_a]_\gamma$$

and, similarly, $[x_\gamma]_\gamma = [x_\beta]_\gamma$, so that $[x_a]_{a \wedge \beta} = [x_\beta]_{a \wedge \beta}$ for all $a, \beta \in A$.

Conversely, suppose that $B \cong \overleftarrow{\mathrm{Lim}}\, B_a$ is a profinite representation of B. Each B_a endowed with the discrete topology is a compact Hausdorff topological Boolean algebra and, therefore, so is the product algebra ΠB_a. $\overleftarrow{\mathrm{Lim}}\, B_a$ being a closed subalgebra of ΠB_a is a compact Hausdorff topological Boolean algebra. Clearly, $\overleftarrow{\mathrm{Lim}}\, B_a$ is order-complete and, therefore (cf. [13]), completely distributive or equivalently atomic.

We remarked that every finite algebra is inversely irreducible. That the converse is true in the class of complete Post algebras is sketched in

THEOREM 5.3. *A complete Post algebra is inversely irreducible if and only if it is finite.*

Proof. Again it suffices to prove the statement of the theorem for Boolean algebras. That there exists at least one proper dense element Δ in the pseudo-Boolean lattice of ideals of any infinite Boolean algebra B follows from the analogue of Theorem 2.6 for Boolean algebras. It is easy to see that 1 is the only upper bound in B of the elements in Δ. Letting Δ play the role of the set $\mathcal{F}$ in the proof of Theorem 5.2, we get easily that B is inversely reducible.

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