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ON COALGEBRAS AND LINEARLY TOPOLOGICAL RINGS

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Introduction. Throughout this note all coalgebras and algebras are over a fixed field k. Radford [16], and Heynemann and Radford [7] elaborate an application of topological methods for studying a coalgebra C by treating its dual algebra C^* as a linearly topological vector space. In [8] Kiełpiński et al. notice that, in fact, C^* is a profinite algebra and following the ideas from Gabriel [6] they show that the relation $C \leftrightarrow C^*$ settles a duality between coalgebras and profinite algebras. Then this duality appears to be a special case of the duality for pseudocompact rings as studied by Brumer [2]. The approach in [8] clears up the ring-topological background of the notion of coreflexivity of a coalgebra, introduced independently by Taft [19] and Radford [16].

In the present note we observe that using, in particular, results from the theory of strictly linearly compact rings, developed by Dieudonné [3], Zelinsky [23], Leptin [9], Müller [13], Warner [20]-[22] and others, one may further clear up the ring-topological nature of such apparently pure coalgebraic notions as locally finite and finite type coalgebras. In particular, it enables us to obtain some new facts for coalgebras, as the equivalence of the following four conditions for a coalgebra C:

(1) C is of finite type,

(2) J-adic and natural profinite topologies on C^* coincide,

(3) J^2 is cofinite,

(4) J is left finitely generated and cofinite.

It settles a duality between coalgebras of finite type and algebras profinite in J-adic topology. An extension of the duality in [8] upon "dense coalgebraic pairings" as defined by Radford [16] is shown, explaining results from [16] in new terms and the Faith method of annihilators [4] is applied to coalgebras, altogether showing a general usefulness of the ring-topological approach to coalgebras. There seems to be no reference of this type in the coalgebraic literature yet.

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1. Preliminaries. Since all basic notions and results on linearly topological rings are easily available in the literature, we assume them known referring to [1], Chapter 2 of [5], [11], Section 2 of [14], or Section 3 of [17]. Here we need the following useful results:

(i) Let R be a left strictly linearly compact ring. Then a left ideal I is open if and only if it is closed and R/I is left Artinian (see [22]).

(ii) For a Hausdorff complete ring R the Jacobson radical J is closed (Proposition 2.9 in [15]).

(iii) For a left linearly compact ring R its Jacobson radical J is open if and only if R/J is Artinian (see [21]).

(iv) If M is Hausdorff left linearly topological module over a left linearly compact ring R, then every finitely generated R-submodule of M is linearly compact and hence closed (see [21]).

LEMMA 1. *Let R be a left strictly linearly compact ring such that*

$$\bigcap_{n\geqslant 0} \overline{J^n} = 0.$$

Then $\overline{J^2}$ is open if and only if the topology of R and the J-adic topology on R coincide.

The lemma follows from (i) and from Theorem 12 in [21].

We say that a ring R is *left almost Noetherian* provided any left ideal I of R is finitely generated whenever R/I is left Artinian (see [7], Section 1).

COROLLARY 1. *Let R be a commutative strictly linearly compact ring such that*

$$\bigcap_{n\geqslant 0} \overline{J^n} = 0.$$

Then the following conditions are equivalent:

(i) *R is Noetherian;*

(ii) *R is almost Noetherian and R/J is Artinian;*

(iii) *the topology on R and the J-adic topology on R coincide.*

For the proof apply Lemma 1, results (i)-(iv), and Theorem 12 of [21] (see also Theorem 7 in [20]).

Remark. We do not know if this equivalence holds in the non-commutative case (**P 1047**). Example 2 in [21] shows that (iii) does not imply (i), it is not clear, however, if the algebra from that example satisfies (ii). All three conditions are equivalent for non-commutative profinite algebras (i.e. complete Hausdorff k-algebras with open two-sided ideals having finite codimension over k) as follows from Theorem 4.1.1 in [7] and from Theorems 1 and 4 of this paper.

2. Profinite duality. Here we gather some facts from [8]; coalgebraic notation and terminology are those from [7], [16] and [18].

Let C be a coalgebra and let R-Mod denote the category of all unitary left modules (rings have units). It is known that rational C^*-modules form a closed subcategory C^*-Dis of C^*-Mod in the sense of [5] or [17], hence there is a "natural" linear ring-topology on C^* for which rational modules are just all discrete C^*-modules (see Proposition 3.3 in [17]). Its preradical is the functor of taking maximal rational submodule $\mathrm{ra}(M)$ of any C^*-module M. Consider a linear topology on C^* induced by the family of all ideals of C of the form

$$D^\perp = \{f \in C^*; f(D) = 0\},$$

where D runs over all finite-dimensional subcoalgebras of C. Since $C^*/D^\perp \approx D^*$ and C is a directed union of finite-dimensional subcoalgebras D_α (Theorem 2.2.1 in [18]), we have $C^* = \lim C^*/D_\alpha^\perp$. Thus C^* is a profinite algebra. We call this topology the *profinite topology* on C^* and denote it by $\mathrm{Pf}(C^*)$.

PROPOSITION 1. *Left discrete C^*-modules over C^* with the profinite topology are just all left rational C^*-modules.*

Proof. Let $m \neq 0$ belong to a rational left C^*-module M. For the associated right comodule structure $\omega\colon M \to M \otimes C$, let

$$\omega(m) = \sum_1^n m_i \otimes c_i$$

and take a finite-dimensional subcoalgebra C_i of C containing c_i for every $i = 1, 2, \ldots, n$. Then

$$(0\colon m) \supset C_1^\perp \cap \ldots \cap C_n^\perp,$$

and so M is discrete (see [17]); conversely, take a discrete C^*-module N. It is enough to show that each cyclic submodule of N is rational. For any $0 \neq n \in N$, since $(0\colon n) \supset D^\perp$ for some finite-dimensional subcoalgebra $D \subset C$, we have

$$C^*/D^\perp \to C^*/(0\colon n) \approx C^* \cdot n.$$

But $C^*/D^\perp \approx D^*$ is rational, and so is $C^* \cdot n$.

It is also known (see [18], p. 109) that there is a functor

$$^\circ\colon \mathrm{Alg}_k \to k\text{-Coalg}$$

from the category of algebras to coalgebras (A° is a subspace of A^* consisting of all linear k-functionals f such that $\mathrm{Ker} f$ contains a two-sided ideal of finite codimension, i.e. $\mathrm{Ker} f$ is open in a cofinite topology on A, denoted by $\mathrm{Cf}(A)$ and induced by such ideals). The above-defined functor is right adjoint to the functor

$$^*\colon k\text{-Coalg} \to \mathrm{Alg}_k.$$

A linearly topological module M over a profinite algebra R is called *profinite* if it is complete Hausdorff and its open submodules have finite codimension (thus it is a special case of pseudocompact modules in the sense of [2] and [5]). Let P_k denote the category of all profinite k-algebras with continuous k-algebra homomorphisms. For any $R \in P_k$ the category of all profinite left (right) R-modules will be denoted by R-Pf (Pf-R). Denote by $\hom(M, N)$ the k-module of continuous homomorphisms in Pf-R. Observe that from Proposition 2.3 in [2] it follows that the functors $^* = \mathrm{Hom}_R(-, k)$ and $S = \hom(-, k)$,

$$R\text{-Dis} \underset{S}{\overset{*}{\rightleftarrows}} \text{Pf-}R,$$

define a duality, which for $R = C^*$ (algebra induced by coalgebra C) is, by Proposition 1, a duality between right C-comodules and right profinite C^*-modules. It is then easy to check that taking $L(A) = S(A_A)$ for $A \in P_k$ we define a functor $L\colon P_k \to k$-Coalg and that we have

THEOREM 1. *The functors*

$$L\colon P_k \to k\text{-Coalg} \quad \textit{and} \quad ^*\colon k\text{-Coalg} \to P_k$$

settle a duality.

Remark. If $\{I_a\}_A$ are open two-sided ideals from a fundamental system of neighbourhoods of zero in $A \in P_k$, then

$$L(A) = \{g \in A^*; \operatorname{Ker} g \supset I_a \text{ for some } a \in A\}.$$

3. Almost profinite algebras and dense pairings. A topological algebra A with fundamental system of neighbourhoods of zero consisting of two-sided ideals, such that the completion of A in this topology is a profinite algebra, is called *almost profinite* (write $A \in AP_k$). For such an algebra A a linearly topological left A-module M is called *almost profinite* (write $M \in A$-APf) if its completion $\hat{M}$ is a profinite $\hat{A}$-module.

LEMMA 2. *For $R \in AP_k$, if N is an open submodule of an almost profinite R-module M, then M/N is finite-dimensional.*

Proof. $\hat{M} \approx \varprojlim M/N$, where N runs over open submodules of M. Let

$$\pi\colon \hat{M} \to M/N \quad \text{and} \quad \varphi\colon M \to \hat{M}$$

be canonical. Define $\Phi\colon M/N \to \hat{M}/\operatorname{Ker}\pi$ by the formula

$$\Phi(m+N) = \varphi(m) + \operatorname{Ker}\pi.$$

From the density of $\varphi(M)$ in $\hat{M}$ it follows that Φ is an isomorphism of discrete modules, which — $\operatorname{Ker}\pi$ being open in a profinite module $\hat{M}$ — completes the proof.

The following is the extension of the profinite duality.

THEOREM 2. *For any almost profinite algebra R there is a diagram with commutative exteriors and interiors:*

$$\begin{tikzcd}
\text{Dis-}R \arrow[r, shift left, "\bar{S}"] \arrow[d, shift left, "\hat{r}"] & R\text{-APf} \arrow[l, shift left, "*"] \arrow[d, shift right, "\hat{}"'] \\
\text{Dis-}\hat{R} \arrow[u, shift left, "\tilde{\varphi}"] \arrow[r, shift left, "*"] & \hat{R}\text{-Pf} \arrow[l, shift left, "S"] \arrow[u, shift right, "\bar{\varphi}"']
\end{tikzcd}$$

such that the rows define dualities and the columns are formed of pairs of adjoint functors defined as follows: $\hat{}$ *is the completion functor,* $r(N) = N$ *treated as* $\hat{R}$*-module for* $N \in \text{Dis-}R$, $\bar{\varphi}$ *and* $\tilde{\varphi}$ *are naturally induced by the canonical*

$$\varphi\colon R \to \hat{R}, \qquad \bar{S} = \hom_R(-, k) \qquad \textit{and} \qquad S = \hom_{\hat{R}}(-, k).$$

Proof. $\hat{r}$ is right adjoint to $\tilde{\varphi}$ and $\hat{}$ is left adjoint to $\bar{\varphi}$ by standard argument. The commutativity claim is obvious, while by the duality over the profinite algebra $\hat{R}$ the rest is straightforward.

Let (C, A) be a ***dense pairing***, i.e. a left non-singular pairing (of vector spaces) of a coalgebra C and of an algebra A such that the induced map $\tau\colon A \to C^*$ is a homomorphism of the algebras (see [16]). For a finite-dimensional subcoalgebra $D \subset C$ write

$$D^{\doteq} = \{a \in A;\ \langle D, a\rangle = 0\}.$$

As in the case of $D^{\perp} \subset C^*$ one checks that $D^{\doteq}$ are two-sided finite-codimensional ideals in A inducing some linear ring-topology on A which we call the ***C-topology*** on A (write $C\text{-top}(A)$). For $A = C^*$ it is just the profinite topology on C^*.

PROPOSITION 2. *For any dense pairing (C, A) the structural algebra map $\tau\colon A \to C^*$ induces an isomorphism of topological algebras $\hat{A} \approx C^*$ (completion in C-topology, C^* with its profinite topology).*

Proof. Notice that, for any finite-dimensional subcoalgebra $D \subset C$, τ induces an isomorphism

$$\bar{\tau}\colon A/D^{\doteq} \xrightarrow{\approx} C^*/D^{\perp}.$$

Now, $\hat{A}$ is profinite for any $A \in \mathbf{AP}_k$ and applying the duality functor L (Theorem 1) we get a coalgebra $L(\hat{A})$ which together with A forms a dense pairing $K(A) = \big(L(\hat{A}), A\big)$ with the structural bilinear form

$$\langle \cdot, \cdot\rangle\colon L(\hat{A}) \times A \to k$$

given by $\langle f, a\rangle = f\big(\varphi(a)\big)$, where $\varphi\colon A \to \hat{A}$ is standard. Thus we have a contravariant functor $K\colon \mathbf{AP}_k \to \text{D-P}$, where D-P denotes the category of dense pairings with morphisms being pairs

$$(C \xrightarrow{f} C',\ A' \xrightarrow{g} A)$$

of coalgebra and algebra maps, respectively, such that

$$\tau \circ g = f^* \circ \tau'$$

($A \xrightarrow{\tau} C^*$ and $A' \xrightarrow{\tau'} C'^*$ are structural); for $h\colon A \to B$ in AP_k we have $K(h) = \big(L(\hat{h}), h\big)$, where $\hat{h}\colon \hat{A} \to \hat{B}$ is naturally induced by h.

Lemma 3. *For any almost profinite algebra A the $L(\hat{A})$-topology on A induced by the dense pairing $K(A)$ coincides with the original topology on A.*

Proof. Let $\{I_a\}_A$ be a set of open two-sided ideals of A inducing the topology of A. Put $D_a = \{f \in A^*;\ \mathrm{Ker} f \supset I_a\}$ and observe that D_a are subcoalgebras in $L(\hat{A}) = L(A)$. In fact, since I_a are two-sided and since coalgebraic Δ in $L(\hat{A})$ is given by the formula $\Delta(f) = f(-, =)$, for $f \in D_a$ we take $g = f(a, =) \in A^*$ ($a \in A$ arbitrary). Since $\mathrm{Ker} g \supset I_a$, by symmetry we have $\Delta(f) \in D_a \otimes D_a$. Now, since I_a is finite-codimensional (by Lemma 2), we get

$$I_a = \bigcap_1^n \mathrm{Ker} f_i \quad \text{for some } f_i \in A^*;$$

thus all f_i are in D_a. If $a \in D_a^{\perp}$, then $\langle D, a\rangle = 0$, whence $\langle f_i, a\rangle = f_i(a) = 0$ which shows that $D_a^{\perp} \subset I_a$; the opposite inclusion is clear by definition.

Notice now that there are two natural functors

$$i\colon k\text{-Coalg} \hookrightarrow \text{D-P},$$

given by $i(C) = (C, C^*)$ with $\langle c, f\rangle = f(c)$, and

$$F\colon \text{D-P} \to k\text{-Coalg},$$

defined by $F(C, A) = C$ with $F(f, g) = f$ for any morphism $(f, g)\colon (C, A) \to (C', A')$ in D-P. It is immediate that F is right adjoint to i.

Finally, we define a contravariant functor $T\colon \text{D-P} \to AP_k$ by $T(C, A)$ being an algebra A with C-topology, and by $T(f, g) = g$ for any morphism (f, g) in D-P. The main fact now is the following

Theorem 3. (i) *The functors K and T settle a duality between the category* D-P *of dense pairings and the category AP_k of almost profinite k-algebras.*

(ii) *In the diagram*

$$\begin{tikzcd}
k\text{-Coalg} \arrow[r, hook, shift left, "i"] \arrow[d, shift right, "*"'] & \text{D-P} \arrow[l, shift left, "F"] \arrow[d, shift left, "T"] \\
P_k \arrow[r, hook, shift right, "j"'] \arrow[u, shift right, "L"'] & AP_k \arrow[l, shift right, "\wedge"'] \arrow[u, shift left, "K"]
\end{tikzcd}$$

where $\hat{}$ *is the completion functor and* j *is a natural embedding, both the interior and the exterior are commutative. After rotating one or both columns the diagram still remains commutative.*

The proof is immediate by using Theorem 1 and Lemma 3.

Remark. An analogous method applied to the category of left non-singular (C, A)-module pairings (in sense of [16]) shows its duality with A-APf extending the case of C-comodules and C^*-Pf.

4. Coreflexivity. Recall that a dense pairing (C, A) is said to be *coreflexive* [16] if the canonical embedding

$$\sigma\colon C \hookrightarrow A^\circ \qquad (\sigma(c)(a) = \langle c, a\rangle)$$

is an isomorphism. A coalgebra C is called *coreflexive* if the pairing (C, C^*) is coreflexive. It is just the coreflexivity of C as defined by Taft [19]. For other definitions see [16]. Given any $(C, A) \in$ D-P with the structural algebra map $\tau\colon A \to C^*$ denote by τ-Cf(A) the linear topology on A induced by $\{\tau^{-1}(I)\}$, where I runs over all two-sided cofinite ideals in C^* (i.e. C^*/I is finite-dimensional), and let Cf(A) be the topology on A induced by all two-sided cofinite ideals of A. Thus there is a sequence of increasingly stronger topologies on A:

$$C\text{-top}(A) \precneqq \tau\text{-Cf}(A) \precneqq \text{Cf}(A).$$

Now we are ready to state two characterizations.

PROPOSITION 3. *If* (C, A) *is a dense pairing, then the following conditions are equivalent:*

(1) (C, A) *is coreflexive,*

(2) C-top$(A) = $ Cf(A),

(3) C *is coreflexive and* τ-Cf$(A) = $ Cf(A).

PROPOSITION 4. *If* (C, A) *is a dense pairing, then the following conditions are equivalent:*

(1) C *is coreflexive,*

(2) Pf$(C^*) = $ Cf(C^*),

(3) C-top$(A) = \tau$-Cf(A),

(4) *for any* $I \in \tau$-Cf(A), $\tau(I)$ *is open in the profinite topology on* C^*.

Remark. Before proving these facts notice that for the cofinite topology on A the class of discrete modules consists precisely of those A-modules for which all cyclic submodules are finite-dimensional, and so from Proposition 3.3 in [17] it follows that C is coreflexive iff all cyclic and, therefore, all finite-dimensional A-modules are rational (see Proposition 8.2 in [19]). Moreover, in Proposition 2.10 of [16] the word "closed" may be replaced by "open", since in an almost profinite algebra an ideal is open iff it is closed and cofinite.

Proof of Proposition 3. (1)$\Rightarrow$(2). For any cofinite ideal I in A we have

$$I = \bigcap_1^n \mathrm{Ker} f_i \quad \text{for some } f_i \in A^*;$$

so $f_i \in A^\circ = \sigma(C)$ and, consequently, $f_i(a) = \langle c_i, a\rangle$ for some $c_i \in C$. If D_i are any finite-dimensional subcoalgebras of C containing c_i, respectively, then $\mathrm{Ker} f_i \supset D_i^{\pm}$, and so

$$I \supset \bigcap_1^n D_i^{\pm}.$$

(2)$\Rightarrow$(1). If $\hat{A}$ is the completion of A in a C-topology, then, by (2),

$$A^\circ = L(\hat{A}) \cong L(C^*) \cong C$$

with last isomorphisms as in Theorem 1 and Proposition 2, altogether giving σ.

(2) $\Leftrightarrow$ (3) follows from Proposition 4.

Proof of Proposition 4. (1)$\Rightarrow$(2) is obvious by (1) and (2) of Proposition 3. It follows from (2) that any cofinite ideal I of C^* contains $D^\perp$, where D is some finite-dimensional subcoalgebra of C, thus $\tau^{-1}(I) > \tau^{-1}(D^\perp) = D^\pm$ and, consequently, $C\text{-top}(A) \geq \tau\text{-Cf}(A)$. Hence these topologies coincide proving (3).

(3)$\Rightarrow$(4)$\Rightarrow$(2) are immediate, since $\tau(D^\pm) = D^\perp$.

Implications (1)$\Leftrightarrow$(2) of Proposition 4 were first proved in [8].

COROLLARY 2. *If a dense pairing (C, A) is coreflexive, then the functor* Ra *of taking the maximal (C, A)-rational submodule of any A-module is a torsion radical.*

Proof. Rational (C, A)-modules are just discrete A-modules, where A is taken with its C-topology. Ra is a preradical for $C\text{-top}(A) = \mathrm{Cf}(A)$, but the cofinite topology preradical is torsion.

We do not know, however, if the reverse is true (**P 1048**). From Corollaries 5 and 7 in [10] and from Theorem 3.4.3 in [7] it is immediate that this problem has a positive answer for a cocommutative coalgebra over algebraically closed field.

5. Artinian coalgebras and coalgebras of finite type. Recall that a coalgebra is said to be *left Artinian* if it has dcc on left coideals. Now we use an idea by Faith [4] to prove the following

PROPOSITION 5. *Let (C, A) be a dense pairing. Then the following conditions are equivalent:*

(1) *C is a left Artinian coalgebra,*

(2) *C^* is a left Noetherian algebra,*

(3) *for every left ideal I in A there is a left ideal $K \subset I$ which is finitely generated and dense in the closure of I.*

Proof. (1)$\Leftrightarrow$(2) is clear by (iv) from Section 1 since the relation $I \leftrightarrow I^{\perp}$ settles the 1-1 correspondence between all left coideals of C and all left ideals of C^* closed in the profinite topology of C^* (see [19]).

(1)$\Leftrightarrow$(3). First observe that C is a right A-module with the multiplication defined by

$$c\cdot a = \sum \langle c_{(1)}, a\rangle \cdot c_{(2)}, \quad \text{where } \Delta(c) = \sum c_{(1)} \otimes c_{(2)}.$$

Since $\varepsilon(c\cdot a) = \langle c, a\rangle$ for any right coideal $I \subset C$, we have

$$\{a \in A;\ I\cdot a = 0\} = I^{\pm}.$$

Now, let $A_r(C, A)$ denote the class of all right ideals in A of the form $I^{\pm}$, where I is a right coideal in C. Then the standard coalgebraic argument shows that $I\cdot a$ are also right coideals in C. Notice also that the class $A_r(C, A)$ has acc iff C is right Artinian. Now it remains to apply Proposition 1 of [4].

Coalgebraic notions such as coradical, coradical filtration, finite type, irreducible and locally finite coalgebras are defined in [7], Section 2.2.

Let $C_0 \subset C_1 \subset \ldots$ be a coradical filtration on a coalgebra C. Recall that for the Jacobson radical J of C^* we have $J = C_0^{\perp}$ and $C_n = (J^{n+1})^{\perp}$ (Proposition 2.1.4 and Corollary 2.1.5 in [7]).

Now, for an arbitrary topological ring R we say that R has the *left special closure property* provided $\overline{I^2}$ is open whenever I is an open left ideal of R. Notice that a coalgebra C is locally finite iff C^* has the special closure property. It is also clear that C is a coalgebra of finite type iff $\overline{J^2}$ is open in $\mathrm{Pf}(C^*)$.

LEMMA 4. *A coalgebra C is almost connected if and only if $C^*/J = C_0^*$ is Artinian.*

Proof. If C_0^* is Artinian, then, by Proposition 5, so is C_0. If there was an infinite set I such that

$$C_0 = \bigoplus_I D_i$$

with D_i distinct simple subcoalgebras of C, then throwing away one D_i after another one could construct a decreasing sequence of subcoalgebras $C_0 \supset E_1 \supset E_2 \supset \ldots$; thus I is finite, which completes the proof.

Now observe that we may extend some results from Sections 2, 4, 5 of [7], avoiding a particular coalgebraic method of reducing general situation to the connected case by the application of "the associated connected coalgebra" (see proofs of Proposition 2.4.3 or Theorem 5.2.1 in [7]).

COROLLARY 3. *Let R be a left strictly linearly compact ring such that*

$$\bigcap_{n\geqslant 0} \overline{J^n} = 0.$$

Then the following conditions are equivalent:

(i) $\overline{J^2}$ *is open,*

(ii) R/J *is left Artinian and* R *has the left special closure property,*

(iii) *the topology on* R *and the* J*-adic topology of* R *coincide,*

(iv) R/J^2 *is left Artinian.*

For the proof apply Theorem 12 from [21] and our Lemma 1.

Combining Corollary 3 with Theorem 1 we get the following

THEOREM 4. *The functors* L *and* $*$ *defined as in Theorem* 1 *settle a duality between all coalgebras of finite type and all algebras profinite in a* J*-adic topology.*

We have also the following

THEOREM 5. *The functors* L *and* $*$ *defined as in Theorem* 1 *settle a duality between all Artinian coalgebras and Noetherian algebras profinite in a* J*-adic topology.*

Proof. If A is a Noetherian algebra profinite in a J-adic topology, then $L(A)$ is an Artinian coalgebra of finite type and the $L(A)$-topology on A is just the J-adic topology by Proposition 5, Corollary 3 and Theorem 4. On the contrary, if C is Artinian, then C^* is Noetherian almost connected, and thus J is open left finitely generated. By Theorem 12 in [21], all J^n are left finitely generated, hence closed, and

$$C^*/J^{n+1} = (\Lambda^n C_0)^*$$

is finite-dimensional since C is locally finite. Thus J^n are open in the profinite topology on C. Now, if D is a finite-dimensional subcoalgebra of C, then $D \subset \Lambda^n C_0$ for some n (Lemma 2.1.3 in [7]), and thus

$$D^{\perp} \supset (\Lambda^n C_0)^{\perp} = \overline{J^{n+1}} \supset J^{n+1},$$

which proves that the J-adic topology and the profinite topology on C coincide.

COROLLARY 4 (see [7]). *If* C *is Artinian, then it is of finite type. If* C *is cocommutative, then* C *is Artinian iff* C *is of finite type.*

Now, for a dense pairing (C, A) with the canonical $\tau: A \to C^*$ we call the *τ-J-adic topology* the topology on A induced by ideals of the form $\tau^{-1}(J^n)$, where J is the Jacobson radical of C.

PROPOSITION 6. *Let* (C, A) *be a dense pairing with the structural map* τ. *Then the following conditions are equivalent:*

(1) C *is of finite type,*

(2) *the* C*-topology and the* τ*-*J*-adic topology on* A *coincide,*

(3) $\tau^{-1}(\overline{J^2})$ *is open in a* C*-topology on* A,

(4) $\tau^{-1}(J^2)$ *is cofinite in* A.

The proof is left to the reader.

Remark. Since C^* is a strictly linearly compact ring, by profinite duality (Theorem 1) it is clear that in a cocommutative case any coalgebra is a direct sum of irreducible coalgebras (possibly infinitely many; see [1], Chapter III, Exercise 21, p. 112). If C is of finite type cocommutative, then it is a finite direct sum of irreducible coalgebras (see Lemma 3 in [20]).

Added in proof. Combining the above remark with Theorem 4.2.6 of [7], Theorem 6 of [10] and Proposition 4.3 of T. Shudo (*A note on coalgebras and rational modules*, Hiroshima Mathematical Journal 6 (1976), p. 297-304) it is easy to solve problem P 1048 positively for a cocommutative coalgebra over an arbitrary field. In a non-cocommutative case the answer is positive for almost connected coalgebras, as follows from our Corollaries 1 and 3, Theorem 4.2.6 of [7] and Theorem 4.6 of Shudo (op. cit.). In particular, it implies that an almost connected coalgebra is of finite type iff the profinite radical ra$(\cdot)$ is torsion. Moreover, combined with results of B. I-Peng Lin (*Semiperfect coalgebras*, Journal of Algebra 49 (1977), p. 357-373), our discussion implies that a cocommutative (or almost connected) coalgebra C which is either (left) semiperfect or projective in (right) C-comodules is also coreflexive.

REFERENCES

[1] N. Bourbaki, *Algèbre commutative*, Paris 1961.

[2] A. Brumer, *Pseudocompact algebras, profinite groups and class formations*, Journal of Algebra 4 (1966), p. 442-470.

[3] J. Dieudonné, *Linearly compact spaces and double vector spaces over s fields*, American Journal of Mathematics 73 (1951), p. 13-19.

[4] C. Faith, *Rings with ascending conditions on annihilators*, Nagoya Mathematical Journal 27 (1966), p. 179-191.

[5] P. Gabriel, *Des catégories abeliennes*, Bulletin de la Société Mathématique de France 90 (1962), p. 323-448.

[6] — *Schémas en groupes*, Séminaire de Géométrie Algébrique 1963-1964, Fascicule 2b, Institut des Hautes Etudes Scientifiques.

[7] R. G. Heynemann and D. E. Radford, *Reflexivity and coalgebras of finite type*, Journal of Algebra 28 (1974), p. 215-246.

[8] R. Kiełpiński, D. Simson and A. Tyc, *On coalgebras and profinite algebras*, Toruń 1973 (unpublished).

[9] H. Leptin, *Linear kompakte Moduln und Ringe*, Mathematische Zeitschrift 62 (1955), p. 241-267.

[10] I. P. Lin, *Products of torsion theories and applications to coalgebras*, Osaka Mathematical Journal 12 (1975), p. 433-439.

[11] I. G. MacDonald, *Duality over complete local rings*, Topology 1 (1962), p. 213-235.

[12] H. Miyamoto, *A note on coreflexive coalgebras*, Hiroshima Mathematical Journal 5 (1975), p. 17-22.

[13] B. J. Müller, *Duality theory for linearly topologized modules*, Mathematische Zeitschrift 119 (1971), p. 63-74.

[14] U. Oberst, *Duality theory for Grothendieck categories and linearly compact rings*, Journal of Algebra 15 (1970), p. 473-542.

[15] — and H. J. Schneider, *Kommutative, F-linear kompakte Ringe*, ibidem 25 (1973), p. 101-149.

[16] D. E. Radford, *Coreflexive coalgebras*, ibidem 26 (1973), p. 512-535.

[17] B. Stenstrom, *Rings and modules of quotients*, Lecture Notes in Mathematics 237 (1971).

[18] M. E. Sweedler, *Hopf algebras*, New York 1969.

[19] E. J. Taft, *Reflexivity of algebras and coalgebras*, American Journal of Mathematics 94 (1972), p. 1111-1130.

[20] S. Warner, *Linearly compact noetherian rings*, Mathematische Annalen 178 (1968), p. 53-61.

[21] — *Linearly compact rings and modules*, ibidem 197 (1972), p. 29-43.

[22] — *Normability of certain topological rings*, Proceedings of the American Mathematical Society 33 (1973), p. 423-427.

[23] D. Zelinsky, *Linearly compact modules and rings*, American Journal of Mathematics 75 (1953), p. 79-90.

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