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EXTENSIONS OF MEASURABLE FUNCTIONS

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1. Introduction. Kuratowski has proved in [2], p. 434, the following basic result on the extension of measurable functions:

Let T be a metric space and X a Polish space. Suppose that $A \subseteq T$ and that $f: A \to X$ is a function of class $a > 0$. Then there are a set A^* of multiplicative class $a+1$ in T and a function $g: A^* \to X$ of class a such that $g = f$ on A. If, moreover, A is assumed to be of multiplicative class a, then there is a function $g: T \to X$ of class a such that $g = f$ on A.

The aim of the present paper is to formulate general selection theorems for closed set-valued maps and to deduce from them results on the extension of measurable functions like the one quoted above. As a result of our formulation we are able to generalize Kuratowski's theorem above to the case where the domain of the function to be extended is a subset of an abstract set and measurability is with respect to fairly general families of subsets of the abstract set.

2. Preliminaries. Let $\mathscr{T}$ be a family of subsets of a set T. We say that $\mathscr{T}$ satisfies the *countable reduction principle* if, whenever $\{A_n: n \geqslant 1\} \subseteq \mathscr{T}$, there exist sets B_n $(n \geqslant 1)$ such that

(a) $(\forall n \geqslant 1)(B_n \in \mathscr{T})$,

(b) $(\forall n \geqslant 1)(B_n \subseteq A_n)$,

(c) $(\forall m, n \geqslant 1)(m \neq n \Rightarrow B_m \cap B_n = \emptyset)$,

(d) $\bigcup_{n\geqslant 1} B_n = \bigcup_{n\geqslant 1} A_n$.

A family $\mathscr{T}$ is said to satisfy the *countable weak reduction principle* if, whenever

$$\{A_n: n \geqslant 1\} \subseteq \mathscr{T} \quad \text{and} \quad \bigcup_{n\geqslant 1} A_n = T,$$

there exist sets B_n $(n \geqslant 1)$ satisfying conditions (a)-(d).

The next lemma provides examples of families satisfying the countable reduction principle. The lemma is known, but we have included the proof here for completeness.

LEMMA 1. *If $\mathscr{L}$ is a field of subsets of a set T, then $\mathscr{L}_\sigma$ satisfies the countable reduction principle.*

Proof. Let $A_n \in \mathscr{L}_\sigma$, $n \geqslant 1$. Write

$$A_n = \bigcup_{k\geqslant 1} A_{nk}, \quad \text{where } A_{nk} \in \mathscr{L},\ n, k \geqslant 1.$$

Set $C_{2^{n-1}(2k-1)} = A_{nk}$, $n, k \geqslant 1$ and put

$$D_1 = C_1 \quad \text{and} \quad D_n = C_n \setminus \bigcup_{i<n} C_i,\ n \geqslant 2.$$

Observe that $D_n \in \mathscr{L}, n \geqslant 1$, $D_n \cap D_m = \emptyset$ for $n \neq m$, and

$$\bigcup_{n\geqslant 1} D_n = \bigcup_{n\geqslant 1} C_n = \bigcup_{n\geqslant 1} \bigcup_{k\geqslant 1} A_{nk}.$$

Finally, putting

$$B_n = \bigcup_{k\geqslant 1} D_{2^{n-1}(2k-1)},$$

we have $B_n \in \mathscr{L}_\sigma$ for $n \geqslant 1$. Since the sets D_n are disjoint, so are the sets B_n. Furthermore,

$$B_n = \bigcup_{k\geqslant 1} D_{2^{n-1}(2k-1)} \subseteq \bigcup_{k\geqslant 1} C_{2^{n-1}(2k-1)} = \bigcup_{k\geqslant 1} A_{nk} = A_n$$

and

$$\bigcup_{n\geqslant 1} B_n = \bigcup_{n\geqslant 1} D_n = \bigcup_{n\geqslant 1} \bigcup_{k\geqslant 1} A_{nk} = \bigcup_{n\geqslant 1} A_n.$$

This completes the proof.

If $\mathscr{T}$ is a family of subsets of a set T and $A \subseteq T$, we denote the family $\{T \setminus B: B \in \mathscr{T}\}$ by $\mathscr{T}^c$ and the family $\{A \cap B: B \in \mathscr{T}\}$ by $\mathscr{T} \cap A$. Assuming that $\mathscr{T}$ is a family of subsets of a set T and $f: T \to X$, where X is a metric space, we say that f is *$\mathscr{T}$-measurable* if $f^{-1}(V) \in \mathscr{T}$ for each open set V in X. In case where T is a metric space and $\mathscr{T}$ the family of subsets of additive class α in T, $\mathscr{T}$-measurable functions on T to X are just the *functions of class α* on T to X. Finally, if $F: T \to 2^X$, 2^X being the space of non-empty closed subsets of X, and if $\mathscr{T}$ is a family of subsets of T, we say that F is *weakly $\mathscr{T}$-measurable* if $\{t \in T: F(t) \cap V \neq \emptyset\} \in \mathscr{T}$ for each open set V in X. The rest of our terminology is from [2].

We need the following result from [3], which we state without proof.

LEMMA 2. *Let $\mathscr{T}$ be a countably additive family of subsets of a set T and let X be a metric space. Let f and f_n $(n \geqslant 1)$ be functions on T to X such that f_n converges uniformly to f. If the functions f_n are $\mathscr{T}$-measurable, then so is f.*

3. Results.

THEOREM 1. *Let Φ be a family of subsets of a set T such that $\emptyset, T \in \Phi$. Assume that Φ is countably additive, finitely multiplicative and satisfies the countable reduction principle. Let X be a Polish space, $A \subseteq T$, and let $F: A \to 2^X$ be weakly $(\Phi \cap A)$-measurable. Then there exist $A^* \in \Phi_\delta$ and a $(\Phi \cap A^*)$-measurable function $g: A^* \to X$ such that*

$$A \subseteq A^* \quad \text{and} \quad (\forall t \in A)\big(g(t) \in F(t)\big).$$

Proof. Fix a complete metric d on X such that diameter of X is less than 1. Let r_n $(n \geqslant 0)$ be an enumeration of a countable dense set in X.

We first define inductively sets A_n and functions $f_n: A_n \to X$ $(n \geqslant 0)$ as follows:

(i) $(\forall n \geqslant 0)(A_n \in \Phi)$,
(ii) $(\forall n \geqslant 1)(A_n \subseteq A_{n-1})$,
(iii) $(\forall n \geqslant 0)(A \subseteq A_n)$,
(iv) $(\forall n \geqslant 0)(\forall t \in A)\big(d(f_n(t), F(t)) < 2^{-n}\big)$,
(v) $(\forall n \geqslant 1)(\forall t \in A_n)\big(d(f_{n-1}(t), f_n(t)) < 2^{-(n-2)}\big)$,
(vi) $(\forall n \geqslant 0)(f_n$ is $(\Phi \cap A_n)$-measurable$)$.

The basic step in the induction is carried out by setting $A_0 = T$ and $f \equiv r_0$ on T. Obviously, A_0 and f_0 satisfy conditions (i)-(vi). Suppose next that A_{n-1} and f_{n-1} satisfy (i)-(vi). For $i \geqslant 1$ put

$$C_i^n = \{t \in A : d(r_i, F(t)) < 2^{-n}\}$$

and

$$D_i^n = \{t \in A_{n-1} : d(r_i, f_{n-1}(t)) < 2^{-(n-2)}\}.$$

Note that $C_i^n \in \Phi \cap A$, so that we can write $C_i^n = E_i^n \cap A$ for sets $E_i^n \in \Phi$. Observe further that by induction hypothesis we have $D_i^n \in \Phi$. Next, for $i \geqslant 1$ set $T_i^n = D_i^n \cap E_i^n$, so that $T_i^n \in \Phi$. Since Φ satisfies the countable reduction principle, there exist sets $S_i^n \in \Phi$ such that $S_i^n \subseteq T_i^n$, $S_i^n \cap S_j^n = \emptyset$ if $i \neq j$ and

$$\bigcup_{i \geqslant 1} S_i^n = \bigcup_{i \geqslant 1} T_i^n.$$

Put

$$A_n = \bigcup_{i \geqslant 1} T_i^n$$

and define $f_n: A_n \to X$ by $f_n(t) = r_i$ for $t \in S_i^n$. We now verify that A_n and f_n have the required properties. Clearly, $A_n \in \Phi$ since Φ is countably additive. Next

$$A_n = \bigcup_{i \geqslant 1} T_i^n \subseteq \bigcup_{i \geqslant 1} D_i^n \subseteq A_{n-1}.$$

To see that $A \subseteq A_n$, let $t \in A$. By induction hypothesis, we have $d(f_{n-1}(t), F(t)) < 2^{-(n-1)}$. Consequently, there is an $x \in F(t)$ such that $d(f_{n-1}(t), x) < 2^{-(n-1)}$. Next choose r_i with $d(x, r_i) < 2^{-n}$. It follows that $d(r_i, F(t)) < 2^{-n}$ and, therefore, $t \in C_i^n \subseteq E_i^n$. Also

$$d(r_i, f_{n-1}(t)) \leqslant d(r_i, x) + d(x, f_{n-1}(t)) < 2^{-n} + 2^{-(n-1)} < 2^{-(n-2)},$$

whence $t \in D_i^n$. Consequently, $t \in D_i^n \cap E_i^n = T_i^n \subseteq A_n$. To verify that f_n satisfies (iv) and (v), consider $t \in A_n$. It follows that $t \in S_i^n$ for some $i \geqslant 1$, so that $f_n(t) = r_i$. Since $S_i^n \subseteq T_i^n$, we have $t \in D_i^n \cap E_i^n$, whence

$$d(f_{n-1}(t), f_n(t)) = d(f_{n-1}(t), r_i) < 2^{-(n-2)}.$$

If also $t \in A$, then

$$d(f_n(t), F(t)) = d(r_i, F(t)) < 2^{-n}.$$

Finally, it follows immediately from the definition of f_n that f_n is $(\Phi \cap A_n)$-measurable. Thus we have checked all conditions (i)-(vi).

Now set

$$A^* = \bigcap_{n \geqslant 1} A_n.$$

Then $A \subseteq A^*$ and $A^* \in \Phi_\delta$. Let g_n be f_n restricted to A^* $(n \geqslant 0)$. Then the functions g_n are $(\Phi \cap A^*)$-measurable and, by virtue of (v) and the completeness of d, g_n converges uniformly to a function $g: A^* \to X$. Now Lemma 2 implies that g is $(\Phi \cap A^*)$-measurable. Since $F(t)$ is closed for each $t \in A$, by (iv) we have $g(t) \in F(t)$ for all $t \in A$. This completes the proof.

THEOREM 2. *Let Φ be a family of subsets of a set T such that $\emptyset, T \in \Phi$. Assume that Φ is countably additive, finitely multiplicative and satisfies the countable weak reduction principle. Let X be a Polish space, $A \in \Phi^c$ and let $F: T \to 2^X$ be weakly Φ-measurable. Assume that $f: A \to X$ is $(\Phi \cap A)$-measurable and that $(\forall t \in A)(f(t) \in F(t))$. Then there is a Φ-measurable function $g: T \to X$ such that*

$$(\forall t \in T)(g(t) \in F(t)) \quad \text{and} \quad (\forall t \in A)(g(t) = f(t)).$$

Proof. Define $G: T \to 2^X$ as follows:

$$G(t) = \begin{cases} \{f(t)\} & \text{for } t \in A, \\ F(t) & \text{for } t \in T \setminus A. \end{cases}$$

To check that G is weakly Φ-measurable assume that V is open in X. Thus

$$\{t \in T: G(t) \cap V \neq \emptyset\} = \{t \in A: f(t) \in V\} \cup \{t \in T \setminus A: F(t) \cap V \neq \emptyset\}.$$

The first set on the right-hand side is in $\Phi \cap A$, and so it can be written as $B \cap A$ for some $B \in \Phi$. The second one can be expressed as $D \cap (T \setminus A)$, where

$$D = \{t \in T: F(t) \cap V \neq \emptyset\} \in \Phi.$$

Since $f(t) \in F(t)$ for each $t \in A$, we have $B \cap A \subseteq D$. Put $C = B \cap D$. Therefore, $C \in \Phi$, $C \subseteq D$ and $C \cap A = B \cap A$. It now follows that

$$\{t \in T: G(t) \cap V \neq \emptyset\} = C \cup (D \cap (T \setminus A)),$$

whence $\{t \in T: G(t) \cap V \neq \emptyset\} \in \Phi$.

Having checked that G is weakly Φ-measurable, it now suffices to use a known selection theorem (Theorem 1 in [4]) to get a Φ-measurable function $g: T \to X$ such that $g(t) \in G(t)$ for each $t \in T$. Then, clearly, $g(t) \in F(t)$ for all $t \in T$ and $g(t) = f(t)$ for all $t \in A$. This completes the proof.

We now use Theorems 1 and 2 to prove results on the extension of measurable functions.

COROLLARY 1. *Let $\mathscr{L}$ be a field of subsets of a set T and let X be a Polish space. Suppose that $A \subseteq T$ and that $f: A \to X$ is $(\mathscr{L}_\sigma \cap A)$-measurable. Then there are a set $A^* \in \mathscr{L}_{\sigma\delta}$ and a function $g: A^* \to X$ such that $A \subseteq A^*$, g is $(\mathscr{L}_\sigma \cap A^*)$-measurable and $g = f$ on A. If, moreover, $A \in (\mathscr{L}_\sigma)^c$, then there is an $\mathscr{L}_\sigma$-measurable function $g: T \to X$ such that $g = f$ on A.*

Proof. Define $F: A \to 2^X$ by $F(t) = \{f(t)\}$. The first assertion now follows from Lemma 1 and Theorem 1.

For the second assertion, define $F: T \to 2^X$ by $F(t) = X$, and then use Lemma 1 and Theorem 2. This completes the proof.

It is to be noted that Kuratowski's theorem, quoted in Section 1, follows from Corollary 1. Indeed, if T is a metric space, one deduces Kuratowski's theorem from Corollary 1 by taking $\mathscr{L}$ to be the family of subsets of T which are simultaneously of additive class a and of multiplicative class a.

The second assertion of Corollary 1 is Corollary 1 in [4].

COROLLARY 2. *Let $\mathscr{L}$ be a field of subsets of a set T and let X be a Polish space. Suppose that $F: T \to 2^X$ is weakly $\mathscr{L}_\sigma$-measurable. Let $A \subseteq T$, let $f: A \to X$ be $(\mathscr{L}_\sigma \cap A)$-measurable and let $f(t) \in F(t)$ for each $t \in A$. Then there is a function $g: T \to X$ such that g is $\mathscr{L}_{\delta\sigma}$-measurable, $g(t) \in F(t)$ for all $t \in T$ and $g = f$ on A.*

Proof. By the first assertion of Corollary 1, there are a set $A^* \in \mathscr{L}_{\sigma\delta}$ and a function $h: A^* \to X$ such that h is $(\mathscr{L}_\sigma \cap A^*)$-measurable, $A \subseteq A^*$ and $h = f$ on A.

Set $B = \{t \in A^*: h(t) \in F(t)\}$. We show next that $B \in \mathscr{L}_{\sigma\delta}$. To see this let r_n $(n \geqslant 1)$ be an enumeration of a countable dense set and let d

be a metric on X. By a simple computation we have

$$B = \{t \in A^*: d(h(t), F(t)) = 0\}$$
$$= \bigcap_{n\geqslant 1} \bigcup_{k\geqslant 1} \left[\left\{t \in A^*: d(r_k, h(t)) < \frac{1}{n}\right\} \cap \left\{t \in A^*: d(r_k, F(t)) < \frac{1}{n}\right\}\right].$$

Now $\{t \in A^*: d(r_k, h(t)) < 1/n\}$ can be expressed as $C_{nk} \cap A^*$ for some set $C_{nk} \in \mathscr{L}_\sigma$, while the set $\{t \in A^*: d(r_k, F(t)) < 1/n\}$ can be written as $D_{nk} \cap A^*$, where

$$D_{nk} = \left\{t \in T: d(r_k, F(t)) < \frac{1}{n}\right\}.$$

Observe that $D_{nk} \in \mathscr{L}_\sigma$. Consequently,

$$B = \Big(\bigcap_{n\geqslant 1} \bigcup_{k\geqslant 1} (C_{nk} \cap D_{nk})\Big) \cap A^*,$$

whence $B \in \mathscr{L}_{\sigma\delta}$. Note also that $A \subseteq B$, since $h(t) = f(t) \in F(t)$ for $t \in A$.

Now let $\Phi = \mathscr{L}_{\delta\sigma}$. Setting $\mathscr{F} = \mathscr{L}_{\delta\sigma} \cap \mathscr{L}_{\sigma\delta}$, we see that $\mathscr{F}$ is a field of subsets of T and that $\Phi = \mathscr{F}_\sigma$. So, by Lemma 1, Φ satisfies the countable reduction principle. Denote by h' the restriction of h to B. Since $\mathscr{L}_\sigma \subseteq \Phi$, h' is $(\Phi \cap B)$-measurable and F is weakly Φ-measurable. Finally, observe that $B \in \mathscr{L}_{\sigma\delta} = \Phi^c$. Theorem 2 now yields a Φ-measurable extension $g: T \to X$ of h' such that $g(t) \in F(t)$ for each $t \in T$. Since $A \subseteq B$, $g(t) = h'(t) = f(t)$ for $t \in A$. This completes the proof.

Himmelberg [1] has proved a special case of Corollary 2, where $\mathscr{L}$ is a σ-field.

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