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THE LOCAL STRUCTURE OF ESSENTIALLY CONFORMALLY SYMMETRIC MANIFOLDS WITH CONSTANT FUNDAMENTAL FUNCTION

III. THE PARABOLIC CASE

BY

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1. Introduction. A Riemannian manifold M of dimension $n \geqslant 4$ (whose metric g may be indefinite) is said to be *conformally symmetric* [1] if its Weyl conformal curvature tensor

$$C_{hijk} = R_{hijk} - (n-2)^{-1}(g_{ij}R_{hk} + g_{hk}R_{ij} - g_{hj}R_{ik} - g_{ik}R_{hj}) + \\ + R(n-1)^{-1}(n-2)^{-1}(g_{ij}g_{hk} - g_{hj}g_{ik}) \tag{1}$$

is parallel, that is

$$C_{hijk,l} = 0. \tag{2}$$

Here and in the sequel we denote by R_{hijk}, R_{ij} and R the curvature tensor, Ricci tensor and scalar curvature, respectively, while the comma stands for covariant differentiation. Clearly, *conformal flatness* ($C_{hijk} = 0$) implies conformal symmetry, and so does *local symmetry* ($R_{hijk,l} = 0$). By *essentially conformally symmetric* (shortly, e.c.s.) Riemannian manifolds we shall mean those which satisfy (2), but are neither conformally flat nor locally symmetric.

E.c.s. manifolds exist ([9], Corollary 3), but their metrics are never definite ([4], Theorem 2). Every e.c.s. manifold satisfies the relation ([6], Theorem 3) $R_{ij}R_{hk} - R_{hj}R_{ik} = FC_{hijk}$ for a certain function F, called the *fundamental function* of M. Two recent papers [2] and [3] are devoted to the e.c.s. manifolds for which $F = \text{const} \neq 0$. In the present paper we treat the so-called *parabolic* e.c.s. manifolds, i.e. those for which $F = 0$ identically, which is clearly equivalent to the condition

$$\operatorname{rank} R_{ij} \leqslant 1 \tag{3}$$

everywhere.

As shown in [9], there exist e.c.s. manifolds which are *Ricci-recurrent* in the sense that $R_{ij}R_{hk,l} = R_{hk}R_{ij,l}$. Such manifolds are always parabolic ([6], Theorem 5). In [9] Roter described the local structure of e.c.s. Ricci-recurrent manifolds. Thus, we shall be mostly concerned with parabolic e.c.s. manifolds which are not Ricci-recurrent (their existence has been established in [6], Theorem 6).

The main result of this paper (Theorem 1) gives a description of the local structure of non-Ricci-recurrent parabolic e.c.s. manifolds. We also show that a non-Ricci-recurrent parabolic e.c.s. manifold is never locally homogeneous (Proposition 1).

In the sequel, all manifolds are assumed to be connected, paracompact and of class C^∞ (although all results remain valid in the analytic category). The methods we use are closely related to those of [2] and [3].

2. Preliminaries. Every e.c.s. manifold satisfies the relations

$$R_{ij,k} = R_{ik,j}, \tag{4}$$

$$R = 0, \tag{5}$$

$$R_{ir}C^r{}_{jkl} = 0, \tag{6}$$

$$R_{hi}C_{jklm} + R_{hj}C_{kilm} + R_{hk}C_{ijlm} = 0 \tag{7}$$

(see [5], Theorems 7, 9 and formula (6), and [6], Theorem 7).

LEMMA 1. *Let M be an n-dimensional Riemannian manifold with a (not necessarily definite) metric g which satisfies* (4) *and* (3). *Given a point $p \in M$ such that*

$$R_{ij}(p) \neq 0, \tag{8}$$

there exists a neighbourhood of p together with two C^∞ vector fields a and b which are uniquely (up to a sign of a) determined by the following two conditions:

$$R_{ij} = \varepsilon a_i a_j, \quad |\varepsilon| = 1, \tag{9}$$

$$R_{ij,k} = b_i R_{jk} + b_j R_{ki} + b_k R_{ij}. \tag{10}$$

Moreover, a and b satisfy the relations $a_i a^i = a_i b^i = 0$ and

$$a_{i,j} = \frac{1}{2} a_i b_j + b_i a_j. \tag{11}$$

Proof. From (3) and (8) we obtain (9) in a neighbourhood of p, $a \neq 0$ being unique up to a sign. From the well-known identity

$$R_i{}^r{}_{,r} = \frac{1}{2} R_{,i} \tag{12}$$

([7], p. 82) it follows by (4) that

(13) $$a_i a^i = \varepsilon R = \text{const},$$

so, by (9),

(14) $$0 = (a_i a_k)_,{}^k = a^k a_{i,k} + a^k{}_{,k} a_i .$$

In view of (4) and (9) we have

(15) $$a_{i,k} a_j + a_i a_{j,k} = \varepsilon R_{ij,k} = \varepsilon R_{ik,j} = a_{i,j} a_k + a_i a_{k,j} .$$

Transvecting this with a^k and using (13), we obtain $\varepsilon R a_{i,j} = a^k a_{i,k} a_j + + a^k a_{j,k} a_i$ (since $a^k a_{k,j} = 0$ by (13)). In view of (14), this yields

(16) $$\varepsilon R a_{i,j} = -2a^k{}_{,k} a_i a_j .$$

Contracting the last equality with g^{ij} and using (13), we obtain $\varepsilon R a^k{}_{,k} = 0$ and, consequently, by (16),

(17) $$a^k{}_{,k} = 0$$

and $R a_{i,j} = 0$, which implies $R = \varepsilon a_i a^i = 0$ (for $R = \text{const} \neq 0$ we have $a_{i,j} = 0$ and, in view of the Ricci identity, $0 = a^i R_{ij} = R a_j$ by (9) and (13)). Consider now, in a neighbourhood of p, the $(n-1)$-plane field $\ker a$ consisting of all vectors u for which $a_i u^i = 0$ (i.e. $u^i R_{ij} = 0$). Given vector fields $u, v \in \ker a$, we have, by the Leibniz rule and (4),

$$v^r u^i{}_{,r} R_{ij} = -v^r u^i R_{ij,r} = -v^r u^i R_{ir,j} = -(v^r u^i R_{ir})_{,j} = 0,$$

so that $D_v u \in \ker a$ (D being the Riemannian connection). Therefore, $[u, v] = D_u v - D_v u \in \ker a$ whenever $u, v \in \ker a$, i.e. $\ker a$ is integrable, which, in terms of the differential 1-form a, can be expressed as

$$a \wedge da = 0$$

or, in the local coordinates,

$$a_i a_{j,k} + a_j a_{k,i} + a_k a_{i,j} - a_i a_{k,j} - a_j a_{i,k} - a_k a_{j,i} = 0 .$$

Subtracting this from

$$a_i a_{j,k} + a_j a_{i,k} - a_k a_{i,j} - a_i a_{k,j} = 0,$$

which is an obvious consequence of (15), we obtain

$$a_j (2a_{i,k} - a_{k,i}) = a_k (2a_{i,j} - a_{j,i}) .$$

Since $a \neq 0$, this yields

$$2a_{j,i} - a_{i,j} = \frac{3}{2} b_j a_i$$

for some vector field b. Alternating this in i and j, we obtain

$$2a_{i,j}-2a_{j,i} = b_i a_j - b_j a_i,$$

so, adding up the last two equalities, we get (11). Using (9), we now obtain (10), which assures the uniqueness of b. Finally, contraction of (11) with g^{ij} yields $a_i b^i = 0$ in view of (17). This completes the proof.

LEMMA 2 ([6], Theorem 4). *If M is a non-Ricci-recurrent e.c.s. manifold, then*

$$C_{hijk} = \eta\omega_{hi}\omega_{jk}, \tag{18}$$

where $|\eta| = 1$ and ω is a parallel, absolute (i.e. determined at each point up to a sign) exterior 2-form on M such that $\operatorname{rank}\omega = 2$ *and* $\omega_{ir}\omega^r{}_j = 0$.

LEMMA 3. *Let M be a non-Ricci-recurrent e.c.s. manifold. Then*

(i) *The image* $\operatorname{im}\omega$ *of ω (the absolute 2-form determined by* (18)), *i.e. the set of all vectors u of type $u_i = \pm\omega_{ij}v^j$, is a parallel field of totally isotropic (2-dimensional) planes and it contains any vector u of the form $u_i = R_{ij}v^j$ or*

$$u_i = R_{ij,k}v_1^j v_2^k. \tag{19}$$

(ii) *The orthogonal complement of* $\operatorname{im}\omega$ *coincides with the kernel* $\ker\omega$ *of ω (the set of all v with $\omega_{ij}v^j = 0$) and it is contained in* $\ker R_{ij}$.

(iii) *The $(n-2)$-plane field* $\ker\omega$ *is integrable and its leaves are totally geodesic submanifolds of M, flat with respect to the symmetric connection they inherit from M* ([8], p. 56-59).

(iv) *The tensor fields R_{ij} and $R_{ij,k}$ are parallel along the leaves of* $\ker\omega$.

Proof. In view of Lemma 3 of [2], all we have to show is that any vector u of form (19) lies in $\operatorname{im}\omega$. However, differentiating (7) covariantly and using (2), (4) and (18), we obtain $u\wedge\omega = 0$, which completes the proof.

In the sequel, we shall often assume the following hypothesis:

(20) (M, g) is an n-dimensional $(n \geqslant 4)$ non-Ricci-recurrent parabolic e.c.s. manifold and $p \in M$ is a point such that

$$R_{ij}(p)R_{hk,l}(p) \neq R_{hk}(p)R_{ij,l}(p). \tag{21}$$

Note that a point p satisfying (21) must exist as M is not Ricci-recurrent.

LEMMA 4. *Under hypothesis* (20), *there exists a C^∞-field a, b of 2-frames in a neighbourhood of p, which is parallel along* $\ker\omega$, *spans* $\operatorname{im}\omega$ (cf. Lemma 3) *and satisfies relations* (9)-(11) *and*

$$b_{i,j} = b_i b_j + 3\lambda b_i a_j + \lambda a_i b_j + \sigma a_i a_j \tag{22}$$

for some C^∞-functions λ and σ. Moreover, we have

$$C_{hijk} = -\Phi(a_h b_i - a_i b_h)(a_j b_k - a_k b_j) \tag{23}$$

for a certain (uniquely determined) function Φ.

Proof. Choose the vector fields a, b as in Lemma 1 (cf. (4) and (21)). In view of (21), (9) and (10), they are linearly independent. Parallelity of a and b along $\ker\omega$ follows immediately from (iv) of Lemma 3. By (i) of Lemma 3, a and b span $\operatorname{im}\omega$, hence ω is a functional multiple of $a\wedge b$, which by (18) yields (23). As b lies in the (parallel) span of a and b and is parallel along its orthogonal complement, $b_{i,j}$ must be a combination of tensor products and squares of a and b. To find the corresponding coefficient functions, we may use the equality $a^r R_{rijk} = 0$, which follows immediately from (1), (6), (5) and (9), and can be rewritten as

$$0 = a_{i,jk} - a_{i,kj} = \frac{1}{2}\, a_i(b_{j,k} - b_{k,j}) + (b_{i,k}a_j - b_{i,j}a_k) + b_i(b_j a_k - b_k a_j).$$

This completes the proof.

Under hypothesis (20), a C^∞-field $c, d, e_3, \ldots, e_{n-2}, b, a$ of n-frames in a neighbourhood of p is called *special* if a and b are the vector fields determined by Lemma 4 and

$$\begin{cases} g(c, e_x) = g(d, e_x) = g(b, e_x) = g(a, e_x) = 0, \\ g(e_x, e_y) = \varepsilon_x \delta_{xy}, \quad |\varepsilon_x| = 1, \quad x, y = 3, \ldots, n-2, \end{cases} \tag{24}$$

$$\begin{cases} g(a, a) = g(a, b) = g(b, b) = 0, \quad g(c, a) = g(d, b) = 1, \\ g(c, c) = g(d, d) = g(c, d) = g(c, b) = g(d, a) = 0, \end{cases} \tag{25}$$

and

$$D_a c = D_a d = D_a e_x = D_b c = D_b d = D_b e_x = D_{e_x} c = D_{e_x} d = D_{e_x} e_y = 0,$$
$$x, y = 3, \ldots, n-2.$$

Here and in the sequel we adopt the convention that the indices x, y, z run over the set $\{3, \ldots, n-2\}$ (empty for $n = 4$).

LEMMA 5. *Assume* (20). *Then*

(i) *There exists a special n-frame field in a neighbourhood of p.*

(ii) *Any special n-frame field $c, d, e_3, \ldots, e_{n-2}, b, a$ in a neighbourhood of p satisfies the relations*

$$D_c c = \xi b - \sigma d - \sum_x \varepsilon_x A_x e_x, \qquad D_c d = -\xi a - c - 3\lambda d - \sum_x \varepsilon_x B_x e_x,$$

$$D_c e_x = A_x a + B_x b + \sum_y C_{xy} e_y, \qquad C_{yx} = -\varepsilon_x \varepsilon_y C_{xy},$$

$$D_c b = 3\lambda b + \sigma a, \qquad D_c a = b,$$

(26)

$$D_d c = \psi b - \frac{1}{2} c - \lambda d - \sum_x \varepsilon_x E_x e_x, \qquad D_d d = -\psi a - d - \sum_x \varepsilon_x F_x e_x,$$

$$D_d e_x = E_x a + F_x b + \sum_y G_{xy} e_y, \qquad G_{yx} = -\varepsilon_x \varepsilon_y G_{xy},$$

$$D_d b = b + \lambda a, \qquad D_d a = \frac{1}{2} a, \qquad D_{e_x} \ldots = D_b \ldots = D_a \ldots = 0,$$

where ... *stands for any frame vector,* λ *and* σ *are determined by* (22), *and* $\xi, \psi, A_x, B_x, E_x, F_x, C_{xy}, G_{xy}$ *are certain* C^∞*-functions. These functions satisfy the relations*

$$(27) \qquad D_a \xi = D_a \sigma = D_a A_x = D_a B_x = D_a C_{xy} = D_a \lambda = D_a \psi = D_a F_x = D_a E_x = D_a G_{xy} = 0,$$

$$(28) \quad \begin{cases} D_b \xi = (n-2)^{-1} \varepsilon, \\ D_b \sigma = D_b A_x = D_b B_x = D_b C_{xy} = D_b \lambda = D_b \psi = 0, \\ D_b F_x = D_b E_x = D_b G_{xy} = 0, \end{cases}$$

$$(29) \quad \begin{cases} D_{e_x} \xi = D_{e_x} \sigma = 0, \qquad D_{e_x} A_y = -(n-2)^{-1} \varepsilon \varepsilon_x \delta_{xy}, \\ D_{e_x} B_y = D_{e_x} C_{yz} = D_{e_x} \lambda = D_{e_x} \psi = D_{e_x} F_y = D_{e_x} E_y = D_{e_x} G_{yz} = 0, \end{cases}$$

$$(30) \quad \begin{cases} D_d A_x - D_c E_x + \lambda B_x - \sigma F_x - 2\lambda E_x + \sum_y (C_{xy} E_y - G_{xy} A_y) = 0, \\ D_d B_x - D_c F_x + \frac{1}{2} B_x - E_x - 5\lambda F_x + \sum_y (C_{xy} F_y - G_{xy} B_y) = 0, \\ D_c G_{xy} - D_d C_{xy} + \frac{1}{2} C_{xy} + 2\lambda G_{xy} + \sum_z (G_{xz} C_{zy} - C_{xz} G_{zy}) = 0, \end{cases}$$

$$(31) \quad \begin{cases} D_d \sigma - D_c \lambda - \sigma + \lambda^2 + (n-2)^{-1} \varepsilon = 0, \qquad D_d \lambda - \frac{3}{2} \lambda = 0, \\ D_d \xi - D_c \psi - 5\lambda\psi + \xi + \sum_x \varepsilon_x (E_x B_x - A_x F_x) = \Phi, \end{cases}$$

Φ *being the function determined by* (23). *Moreover, we have*

$$(32) \qquad D_c \Phi = -6\lambda\Phi, \qquad D_d \Phi = -3\Phi, \qquad D_{e_x} \Phi = D_b \Phi = D_a \Phi = 0.$$

Proof. According to Lemma 4, the vector fields a and b determined by (9) and (10) span $\operatorname{im} \omega$ and are parallel along $\ker \omega$, so that the existence of a special n-frame field is an immediate consequence of Lemma 6 of [2].

Now let $c, d, e_3, \ldots, e_{n-2}, b, a$ be a special n-frame field in a neighbourhood of p. Relations (26) follow immediately from (11), (22), (24) and (25) by means of the Leibniz rule. Moreover, the only non-trivial components of R_{hijk} with respect to our frame are, in view of (1), (5), (9) and (23), those related to

$$R_{hijk}c^h d^i c^j d^k = -\Phi, \qquad R_{hijk}b^h c^i c^j d^k = (n-2)^{-1}\varepsilon,$$
$$R_{hijk}c^h e_x^i c^j e_x^k = -(n-2)^{-1}\varepsilon\varepsilon_x.$$

On the other hand, we can find these components from the formulae

$$R(u, v)w = D_u D_v w - D_v D_u w - D_{[u,v]}w$$

and

(33) $$[u, v] = D_u v - D_v u.$$

Thus, (27) can be obtained from computing $R_{acc} = R(a, c)c$, $R_{acx} = R(a, c)e_x$, R_{adb}, R_{add} and R_{adx}, (28) from $R_{bcc} = (n-2)^{-1}\varepsilon b$, R_{bcx}, R_{bdb}, R_{bdd} and R_{bdx}, (29) from $R_{cxc} = -(n-2)^{-1}\varepsilon e_x$, $R_{cxy} = (n-2)^{-1}\varepsilon\varepsilon_x\delta_{xy}$, R_{dxb}, R_{dxd} and R_{dxy}, (30) from R_{cdx}, (31) from $R_{cdb} = (n-2)^{-1}\varepsilon a$ and $R_{cdc} = -\Phi b-(n-2)^{-1}\varepsilon d$. Finally, (32) follows from the obvious (in view of (23) and (25)) relation $C_{hijk}c^h d^i c^j d^k = -\Phi$ together with (2) and (26). This completes the proof.

A Riemannian manifold M is said to be *locally homogeneous* if for any two points $p, q \in M$ there exists an isometry of a neighbourhood of p onto a neighbourhood of q, sending p onto q.

PROPOSITION 1. *Let M be a non-Ricci-recurrent parabolic e.c.s. manifold. Then M is not locally homogeneous.*

Proof. If M were locally homogeneous, then (23) would imply $\Phi = \text{const}$ (in view of (9) and (10), any local isometry leaves $\pm a$ and b invariant). This, together with (32), would yield $\Phi = 0$ (as Lemma 5 works), a contradiction, which completes the proof.

LEMMA 6. *Under hypothesis* (20), *there exists a special n-frame field $c, d, e_3, \ldots, e_{n-2}, b, a$ in a neighbourhood of p such that, in the notation of Lemma* 5,

(34) $$C_{xy} = G_{xy} = 0,$$

(35) $$E_x = F_x = 0,$$

(36) $$B_x = D_d A_x = 0,$$

(37) $$\psi = 0,$$

(38) $$D_c(D_c\xi + 3\lambda\xi) = \sigma\xi - \frac{1}{2}\sum_x \varepsilon_x A_x^2 + \Phi\left(\frac{3}{2}D_c\lambda - 5\lambda^2 + \frac{1}{2}\sigma - \frac{1}{6}(n-2)^{-1}\varepsilon\right).$$

Proof. 1st step. In view of Lemma 5, we may choose a special n-frame field $c, d, e_3, \ldots, e_{n-2}, b, a$ in a neighbourhood of p. The integrability conditions for the system of differential equations

$$(39)\qquad \begin{gathered} D_c\tau_{xy} = -\sum_z \tau_{xz}C_{zy}, \quad D_d\tau_{xy} = -\sum_z \tau_{xz}G_{zy}, \\ D_{e_z}\tau_{xy} = D_b\tau_{xy} = D_a\tau_{xy} = 0, \end{gathered}$$

with indeterminates τ_{xy} follow immediately from (27)-(30). Choosing the solution τ_{xy} of (39) with initial value $\tau_{xy}(p) = \delta_{xy}$, it is easy to verify (by differentiation) that

$$\sum_z \varepsilon_z\tau_{xz}\tau_{yz} = \varepsilon_x\delta_{xy}.$$

Therefore, the n-frame field

$$c, d, \bar{e}_3, \ldots, \bar{e}_{n-2}, b, a, \quad \text{where } \bar{e}_x = \sum_y \tau_{xy}e_y,$$

is special and satisfies (34).

2nd step. Let $c, d, e_3, \ldots, e_{n-2}, b, a$ be a special n-frame field satisfying (34). Consider the underdetermined system of differential equations

$$(40)\qquad \begin{gathered} D_d\zeta_x = -\frac{1}{2}\zeta_x - \lambda\iota_x - E_x, \quad D_{e_y}\zeta_x = D_b\zeta_x = D_a\zeta_x = 0, \\ D_d\iota_x = -\iota_x - F_x, \quad D_{e_y}\iota_x = D_b\iota_x = D_a\iota_x = 0 \end{gathered}$$

with unknown functions ζ_x and ι_x. Since the vector fields $d, e_3, \ldots, e_{n-2}, b, a$ span an involutive distribution (which follows immediately from (26) and (33)), it is clear that a solution of (40) will exist if its integrability conditions (which do not involve differentiation along c) are satisfied. However, these conditions follow immediately from (27)-(29). Choosing a solution ζ_x, ι_x of (40) and setting

$$\bar{c} = c - \sum_x \varepsilon_x\zeta_x e_x - \frac{1}{2}\sum_x \varepsilon_x\zeta_x^2 a,$$

$$\bar{d} = d - \sum_x \varepsilon_x\iota_x e_x - \frac{1}{2}\sum_x \varepsilon_x\iota_x^2 b - \sum_x \varepsilon_x\zeta_x\iota_x a, \qquad \bar{e}_x = e_x + \zeta_x a + \iota_x b,$$

it is easy to verify that $\bar{c}, \bar{d}, \bar{e}_3, \ldots, \bar{e}_{n-2}, b, a$ is a special n-frame field satisfying (34) and (35).

3rd step. Choose a special n-frame field $c, d, e_3, \ldots, e_{n-2}, b, a$ satisfying (34) and (35) and set, in the notation of Lemma 5,

$$\bar{c} = c + \sum_x \varepsilon_x B_x e_x - \frac{1}{2}\sum_x \varepsilon_x B_x^2 a, \qquad \bar{e}_x = e_x - B_x a.$$

Using (27)-(30) it is easy to see that $\bar{c}, d, \bar{e}_3, \ldots, \bar{e}_{n-2}, b, a$ is a special n-frame field satisfying (34), (35) and $B_x = 0$. The relation $D_d A_x = 0$ follows now immediately from (30).

4th step. Let a special n-frame field $c, d, e_3, \ldots, e_{n-2}, b, a$ satisfy (34)-(36) and set $\bar{c} = c - hb$, $\bar{d} = d + ha$, where h is any solution of the underdetermined system

$$D_d h = \psi - \frac{3}{2} h, \quad D_{e_x} h = D_b h = D_a h = 0$$

(completely integrable in view of (27)-(29)). It is now easy to verify that $\bar{c}, \bar{d}, e_3, \ldots, e_{n-2}, b, a$ satisfies (34)-(37).

5th step. Let $c, d, e_3, \ldots, e_{n-2}, b, a$ be a special n-frame field satisfying (34)-(37). Using Lemma 5 and (33), it is easy to verify that, in the notation of Lemma 5,

$$D_a D_c \lambda = D_b D_c \lambda = D_{e_x} D_c \lambda = 0, \quad D_d D_c \lambda = 2D_c \lambda + 3\lambda^2,$$
$$D_a D_c D_c \lambda = D_b D_c D_c \lambda = D_{e_x} D_c D_c \lambda = 0,$$
$$D_d D_c D_c \lambda = \frac{5}{2} D_c D_c \lambda + 10\lambda D_c \lambda + 6\lambda^3,$$

$$D_a D_c \xi = -(n-2)^{-1}\varepsilon, \quad D_b D_c \xi = -3(n-2)^{-1}\varepsilon\lambda, \quad D_{e_x} D_c \xi = 0,$$
$$D_d D_c \xi = -\frac{1}{2} D_c \xi - 4\lambda D_d \xi - 6\lambda\xi, \quad D_a D_d \xi = D_{e_x} D_d \xi = 0,$$

(41)

$$D_b D_d \xi = -(n-2)^{-1}\varepsilon, \quad D_c D_d \xi = -D_c \xi - 6\lambda D_d \xi - 6\lambda\xi,$$
$$D_d D_d \xi = -4D_d \xi - 3\xi, \quad D_a D_c D_c \xi = 3(n-2)^{-1}\varepsilon\lambda,$$
$$D_{e_x} D_c D_c \xi = (n-2)^{-1}\varepsilon A_x,$$
$$D_b D_c D_c = (n-2)^{-1}\varepsilon\sigma + 9(n-2)^{-1}\varepsilon\lambda^2 - 3(n-2)^{-1}\varepsilon D_c \lambda,$$
$$D_d D_c D_c = 12\lambda^2 \xi - 3\lambda D_c \xi - 4D_c \lambda \cdot D_d \xi + 16\lambda^2 D_d \xi - 6\xi D_c \lambda - (n-2)^{-1}\varepsilon\xi,$$

$$D_a D_c A_x = D_b D_c A_x = D_{e_y} D_c A_x = 0, \quad D_d D_c A_x = \frac{1}{2} D_c A_x.$$

Set

$$Q = D_c D_c \xi + 3\lambda D_c \xi + 3\xi D_c \lambda - \sigma\xi + \frac{1}{2} \sum_x \varepsilon_x A_x^2 -$$
$$- \Phi\left(\frac{3}{2} D_c \lambda - 5\lambda^2 + \frac{1}{2}\sigma - \frac{1}{6}(n-2)^{-1}\varepsilon\right).$$

From (41) and Lemma 5 we obtain $D_aQ = D_bQ = D_{e_x}Q = D_dQ = 0$. Consider now the system of differential equations

$$D_c h = a, \quad D_d h = -\frac{3}{2}h, \quad D_{e_x}h = D_b h = D_a h = 0,$$

$$D_c a = \beta, \quad D_d a = -a - 3\lambda h, \quad D_{e_x}a = D_b a = D_a a = 0,$$

(42)

$$D_c\beta = Q - 3hD_cD_c\lambda - 9aD_c\lambda - 6\lambda\beta - 18\lambda hD_c\lambda - 9\lambda^2 a + \\ + \sigma a + 3\lambda\sigma h - (n-2)^{-1}\varepsilon a,$$

$$D_d\beta = -\frac{1}{2}\beta - 5\lambda a - 3hD_c\lambda - 6\lambda^2 h, \quad D_{e_x}\beta = D_b\beta = D_a\beta = 0$$

with unknown functions h, a and β. Its integrability conditions are immediate consequences of (26)-(29), (32) and (31). For instance, we have

$$D_cD_d\beta - D_dD_c\beta - D_{[c,d]}\beta = -(a + 3\lambda h)\big(D_d\sigma - D_c\lambda - \sigma + \lambda^2 + (n-2)^{-1}\varepsilon\big),$$

which vanishes in virtue of (31), etc. Choose now a solution h, a, β of (42). It is easy to verify that the n-frame field $\bar{c}, \bar{d}, e_3, \ldots, e_{n-2}, b, a$, where $\bar{c} = c - hb$ and $\bar{d} = d + ha$, satisfies our assertion. This completes the proof.

3. Local structure theorem. We are now in a position to prove our main result.

THEOREM 1. (i) *Let g be the indefinite metric on an open subset U of R^n ($n \geqslant 4$) whose non-zero components at any point $(u^1, \ldots, u^n)$ of U are given by*

(43)

$$g_{11} = \sum_x \varepsilon_x P_x^2 - 2(n-2)\delta\varepsilon e^{-4T}\varphi'' - (n-2)\delta\varepsilon e^{-6T}\tau + \frac{2}{3}\delta e^{-8T} - \\ -(n-2)^{-1}\varepsilon e^{-2T}\sum_x \varepsilon_x (u^x)^2 + \\ + u^{n-1}\left[\frac{1}{2}e^{4T}(\varphi')^2 - e^{2T}\varphi'' - 2\tau + 2(n-2)^{-1}\varepsilon e^{-2T}\right],$$

$$g_{12} = g_{21} = -2u^{n-1}\varphi' e^{4T} + \frac{3}{2}u^n e^T + \frac{1}{2}(n-2)\delta\varepsilon\varphi' e^{-2T},$$

$$g_{22} = -2(n-2)\delta\varepsilon e^{-2T} + 2u^{n-1}e^{4T},$$

$$g_{1x} = g_{x1} = P_x(u^1), \quad g_{xx} = \varepsilon_x,$$

$$g_{1n} = g_{n1} = e^{-T}, \quad g_{2,n-1} = g_{n-1,2} = e^{2T},$$

where

$$(44) \qquad |\delta| = |\varepsilon| = |\varepsilon_x| = 1,$$

and P_x, τ *and* φ *are functions depending only on the first variable* u^1 *and such that*

$$(45) \qquad \varphi(u^1)-u^2 > 0$$

for any $(u^1, \ldots, u^n) \in U$, *while* T *is defined by*

$$(46) \qquad T(u^1, u^2) = -\frac{1}{2}\log\big(\varphi(u^1)-u^2\big).$$

Then g *is e.c.s., parabolic and non-Ricci-recurrent (more precisely,* (21) *holds at each point* $p \in U$).

(ii) *Conversely, given an n-dimensional* ($n \geqslant 4$) *parabolic e.c.s. non-Ricci-recurrent manifold* (M, g) *and a point* $p \in M$ *satisfying* (21), *there exists a local coordinate system* $u^1, \ldots, u^n$ *in a neighbourhood of* p *such that the components of* g *are given by* (43), (44) *and* (46), *where* P_x, τ *and* φ *are functions of* u^1 *satisfying* (45) *in a neighbourhood of* p.

Proof. (i) The non-zero contravariant components of our metric are

$$g^{1n} = e^T, \quad g^{2,n-1} = e^{-2T}, \quad g^{xx} = \varepsilon_x, \quad g^{xn} = -e^T\varepsilon_x P_x,$$
$$g^{n-1,n-1} = -e^{-4T}g_{22}, \quad g^{n-1,n} = -e^{-T}g_{12}, \quad g^{nn} = e^{2T}\Big(\sum_x \varepsilon_x P_x^2 - g_{11}\Big).$$

We can now compute the following components of the Riemannian connection:

$$\Gamma^1_{11} = \Gamma^2_{12} = \frac{1}{2}\varphi' e^{2T}, \quad \Gamma^1_{12} = -e^{2T}, \quad \Gamma^1_{22} = \Gamma^2_{22} = 0,$$

$$\Gamma^2_{11} = \frac{1}{2}\varphi'' - \frac{1}{4}(\varphi')^2 e^{2T} + \tau e^{-2T} - (n-2)^{-1}\varepsilon e^{-4T},$$

$$\Gamma^x_{11} = \varepsilon_x P_x' + (n-2)^{-1}\varepsilon e^{-2T}u^x - \frac{1}{2}\varepsilon_x P_x e^{2T}\varphi', \quad \Gamma^x_{12} = e^{2T}\varepsilon_x P_x, \quad \Gamma^x_{22} = 0,$$

$$(47) \qquad \Gamma^1_{ij} = \Gamma^2_{ij} = \Gamma^x_{ij} = 0 \quad \text{if } i > 2 \text{ or } j > 2,$$

$$\Gamma^{n-1}_{11} = u^{n-1}\Big[5(\varphi')^2 e^{4T} - \frac{5}{2}\varphi'' e^{2T} - 2\tau + 3(n-2)^{-1}\varepsilon e^{-2T}\Big] -$$
$$-\frac{3}{2}u^n\varphi' e^T - \frac{1}{2}(n-2)^{-1}\varepsilon e^{-2T}\sum_x \varepsilon_x (u^x)^2 - \frac{1}{4}(n-2)\delta\varepsilon(\varphi')^2 e^{-2T} -$$
$$-\frac{1}{2}(n-2)\delta\varepsilon\varphi'' e^{-4T} + \frac{1}{2}(n-2)\delta\varepsilon\tau e^{-6T} - \frac{2}{3}\delta e^{-8T},$$

$$\Gamma_{12}^{n-1} = -5u^{n-1}\varphi' e^{4T} + \frac{3}{2}u^n e^T + \frac{1}{2}(n-2)\delta\varepsilon\varphi' e^{-2T},$$

$$\Gamma_{22}^{n-1} = 2u^{n-1}e^{4T} + (n-2)\delta\varepsilon e^{-2T}, \quad \Gamma_{1x}^{n-1} = \Gamma_{2x}^{n-1} = 0,$$

$$\Gamma_{1,n-1}^{n-1} = -\frac{3}{2}e^{2T}\varphi', \quad \Gamma_{1n}^{n-1} = e^{-T}, \quad \Gamma_{2,n-1}^{n-1} = e^{2T}, \quad \Gamma_{2n}^{n-1} = 0,$$

$$\Gamma_{12}^{n} = u^{n-1}\left[2(\varphi')^2 e^{7T} - \frac{3}{2}\varphi'' e^{5T} + (n-2)^{-1}\varepsilon e^T - 2\tau e^{3T}\right] -$$

$$-\frac{3}{4}u^n\varphi' e^{4T} - \frac{1}{2}(n-2)^{-1}\varepsilon e^T \sum_x \varepsilon_x (u^x)^2 - \frac{2}{3}\delta e^{-5T} + \frac{1}{2}(n-2)\delta\varepsilon\tau e^{-3T} -$$

$$-\frac{1}{4}(n-2)\delta\varepsilon(\varphi')^2 e^T.$$

It is easy to verify that

$$R_{121}{}^1 = 0, \quad R_{121}{}^2 = -(n-2)^{-1}\varepsilon e^{-2T} \quad \text{and} \quad R_{121}{}^{n-1} = -\delta e^{-6T}.$$

From (47) it follows immediately that $R_{ijk}{}^l = 0$ whenever $l \leqslant n-2$ and $k \geqslant 3$, which implies $R_{ijkl} = 0$ if $k, l \geqslant 3$. Therefore, the only non-zero components of the curvature tensor, Ricci tensor and Weyl's tensor are related to

$$R_{1212} = \delta e^{-4T} - 2(n-2)^{-1}\varepsilon u^{n-1}e^{2T}, \quad R_{121,n-1} = -(n-2)^{-1}\varepsilon,$$

$$R_{1x1x} = -(n-2)^{-1}\varepsilon\varepsilon_x e^{-2T} \quad \text{and} \quad R_{11} = \varepsilon e^{-2T}, \quad C_{1212} = -\delta e^{-4T}.$$

It is now a trivial matter to check that $C_{hijk,l} = 0$. Moreover, $R_{11}R_{12,1} - R_{12}R_{11,1} = e^{-2T}$, which completes the proof of (i).

(ii) Using the notation of Lemma 5, choose a special n-frame field $c, d, e_3, \ldots, e_{n-2}, b, a$ in a neighbourhood of p, which satisfies (34)-(38) (cf. Lemma 6). Define the function T by

$$\Phi = \delta e^{-6T}, \quad |\delta| = 1, \tag{48}$$

so that, by (32),

$$D_c T = \lambda, \quad D_d T = \frac{1}{2}, \quad D_{e_x} T = D_b T = D_a T = 0. \tag{49}$$

Therefore, and in view of (26), the systems of differential equations

$$D_c u^1 = e^T, \quad D_d u^1 = D_{e_x} u^1 = D_b u^1 = D_a u^1 = 0$$

and

$$D_d u^2 = e^{-2T}, \quad D_c u^2 = D_{e_x} u^2 = D_b u^2 = D_a u^2 = 0,$$

with unknown functions u^1, u^2, are completely integrable. Choosing their solutions u^1, u^2 and setting

$$u^x = -(n-2)\varepsilon\varepsilon_x A_x, \quad u^{n-1} = (n-2)\varepsilon\xi, \qquad u^n = -(n-2)\varepsilon(D_c\xi+3\lambda\xi), \tag{50}$$

we obtain a local coordinate system $u^1, \ldots, u^n$ in a neighbourhood of p. In fact, the vector fields ∂_i dual to $du^1, \ldots, du^n$ are given by

$$\partial_1 = e^{-T}c+(n-2)\varepsilon e^{-T}\sum_x \varepsilon_x D_c A_x e_x-(n-2)\varepsilon e^{-T}D_c\xi\cdot b+$$
$$+(n-2)\varepsilon e^{-T}\Big[\sigma\xi-\frac{1}{2}\sum_x \varepsilon_x A_x^2+\Phi\Big(\frac{3}{2}D_c\lambda-5\lambda^2+\frac{1}{2}\sigma-\frac{1}{6}(n-2)^{-1}\varepsilon\Big)\Big]a,$$

(51)

$$\partial_2 = e^{2T}d-(n-2)\varepsilon e^{2T}D_d\xi\cdot b-\frac{1}{2}(n-2)\varepsilon e^{2T}(D_c\xi+3\lambda\xi+2\lambda D_d\xi)a,$$

$$\partial_x = e_x, \quad \partial_{n-1} = b, \quad \partial_n = a,$$

which follows immediately from (27)-(29), (38), (41), (36) and (31). Now (49) and (51) yield

$$\partial_n e^{-2T} = \partial_{n-1}e^{-2T} = \partial_x e^{-2T} = 0, \quad \partial_2 e^{-2T} = -1,$$

which implies (46) for some function φ of u^1, satisfying (45). From (49) we obtain

$$\lambda = D_c T = \partial_1 e^T = -\frac{1}{2}\varphi'(u^1)e^{3T}. \tag{52}$$

Moreover, by (50),

$$D_c\xi = -(n-2)^{-1}\varepsilon u^n+\frac{3}{2}(n-2)^{-1}\varepsilon u^{n-1}\varphi' e^{3T} \tag{53}$$

and, by (37), (35), (48), (31) and (50),

$$D_d\xi = \delta e^{-6T}-(n-2)^{-1}\varepsilon u^{n-1} \tag{54}$$

and, in view of (49),

$$D_c\lambda = -\frac{1}{2}\varphi'' e^{4T}+\frac{3}{4}(\varphi')^2 e^{6T}. \tag{55}$$

On the other hand, (41) yields

$$\partial_n D_c A_x = \partial_{n-1}D_c A_x = \partial_y D_c A_x = 0, \quad \partial_2 D_c A_x = \partial_2 T\cdot D_c A_x,$$

whence

$$D_c A_x = (n-2)^{-1} \varepsilon P_x(u^1) e^T \tag{56}$$

for some functions P_x. Setting

$$\tau = e^{-2T}[D_c\lambda - \sigma - 2\lambda^2 + (n-2)^{-1}\varepsilon],$$

in view of (41), (27)-(29), (31) and (49) we obtain

$$D_d\tau = D_{e_x}\tau = D_b\tau = D_a\tau = 0,$$

which, in terms of our chart, states that τ is a function only of u^1. Thus

$$\sigma = D_c\lambda - \tau(u^1)e^{2T} - 2\lambda^2 + (n-2)^{-1}\varepsilon. \tag{57}$$

Computing now the components $g_{ij} = g(\partial_i, \partial_j)$ of the metric from (51), (24), (25) and (50) and (52)-(57), we obtain (43). This completes the proof.

REFERENCES

[1] M. C. Chaki and B. Gupta, *On conformally symmetric spaces*, Indian Journal of Mathematics 5 (1963), p. 113-122.

[2] A. Derdziński, *The local structure of essentially conformally symmetric manifolds with constant fundamental function, I. The elliptic case*, Colloquium Mathematicum 42 (1979), p. 59-81.

[3] — *The local structure of essentially conformally symmetric manifolds with constant fundamental function, II. The hyperbolic case*, ibidem 44 (1981), p. 77-95.

[4] — and W. Roter, *On conformally symmetric manifolds with metrics of indices* 0 *and* 1, Tensor, New Series, 31 (1977), p. 255-259.

[5] — *Some theorems on conformally symmetric manifolds*, ibidem 32 (1978), p. 11-23.

[6] — *Some properties of conformally symmetric manifolds which are not Ricci-recurrent*, ibidem 34 (1980), p. 11-20.

[7] L. P. Eisenhart, *Riemannian geometry*, Princeton 1949.

[8] S. Kobayashi and K. Nomizu, *Foundations of differential geometry*, Vol. II, New York 1969.

[9] W. Roter, *On conformally symmetric Ricci-recurrent spaces*, Colloquium Mathematicum 31 (1974), p. 87-96.

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