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ON CONFORMALLY SYMMETRIC 2-RICCI-RECURRENT SPACES

BY

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1. Introduction. An n-dimensional Riemannian space is said to be *of recurrent curvature* (cf. [8] and [13]) if its curvature tensor satisfies, for some vector c_j, the condition

$$R_{hijk,l} = c_l R_{hijk},$$

where the comma indicates covariant differentiation with respect to the metric of the space.

In generalizing this concept Lichnerowicz [3] has called a Riemannian n-space, whose curvature tensor satisfies

$$R_{hijk,lm} = a_{lm} R_{hijk} \tag{1}$$

for some tensor a_{ij}, a *second-order recurrent space* (briefly, *2-recurrent space*).

Spaces of such a type, i.e. satisfying (1) for $a_{ij} \neq 0$, have been considered mainly by Thompson [9]-[11].

Investigating conformally flat 2-recurrent spaces, Roy Chowdhury discovered [6] that the trace of a_{ij} vanishes and the scalar curvature of the space is zero. Using this result, Thompson was able to prove [12] that every conformally flat 2-recurrent space reduces to a space of recurrent curvature.

As a generalization of the concept of a second-order recurrent space, Roy Chowdhury [7] initiated investigations of n-dimensional Riemannian spaces whose Ricci-tensors satisfy relation of the form

$$R_{ij,lm} = a_{lm} R_{ij} \tag{2}$$

for some tensor a_{ij}. Spaces of this kind, called *second-order Ricci-recurrent* or, briefly, *2-Ricci-recurrent spaces*, are evidently generalizations of so called *Ricci-recurrent spaces* [4], i.e., of Riemannian n-spaces characterized by the condition

$$R_{ij,l} = c_l R_{ij}. \tag{3}$$

According to Chaki and Gupta [2], an n-dimensional ($n > 3$) Riemannian space is called *conformally symmetric* if its Weyl's conformal tensor

$$(4) \qquad C^h{}_{ijk} = R^h{}_{ijk} - \frac{1}{n-2}(g_{ij}R^h{}_k - g_{ik}R^h{}_j + \delta^h_k R_{ij} - \delta^h_j R_{ik}) + \\ + \frac{R}{(n-1)(n-2)}(\delta^h_k g_{ij} - \delta^h_j g_{ik})$$

satisfies

$$(5) \qquad C^h{}_{ijk,l} = 0 .$$

It follows easily from (4) and (5) that every conformally flat as well as every symmetric (in the sense of E. Cartan) Riemannian n-space ($n > 3$) is necessarily conformally symmetric. Converse of this, as we shall show in a subsequent paper, is in general not true.

Conformally symmetric Ricci-recurrent spaces have been studied by Adati and Miyazawa [1]. Their main result is the following:

If the Ricci-tensor of a conformally symmetric space satisfies (3) for a non-zero vector c_j, then the following cases occur:

(a) the space is conformally flat and recurrent (of recurrent curvature),

(b) the space is symmetric in the sense of Cartan and $R_{ij} = 0$,

(c) the scalar curvature vanishes and the recurrence vector c_j is null.

The present author established [5] that if the Ricci-tensor of a conformally symmetric Ricci-recurrent space with non-vanishing vector c_j is assumed to be non-zero, then the scalar curvature of the space vanishes and the vector of recurrence c_j is null.

The purpose of this paper is to obtain an analogical result for conformally symmetric 2-Ricci-recurrent spaces. Namely, continuing Roy Chowdhury's investigations, it will be proved that a conformally symmetric 2-Ricci-recurrent space reduces to a Ricci-recurrent space with the necessarily vanishing scalar curvature.

We emphasize that our definition of a 2-Ricci-recurrent space assumes $a_{ij} \neq 0 \neq R_{ij}$ at every point.

2. Preliminary results. First we shall obtain a result on general conformally symmetric spaces:

LEMMA 1. *The curvature tensor of a conformally symmetric space satisfies the equation*

$$(6) \qquad R_{rj}R^r{}_{ikl} + R_{rk}R^r{}_{ilj} + R_{rl}R^r{}_{ijk} = 0 .$$

Proof. Differentiating (4) covariantly, summing for h, l and taking into account (5) and the well-known formulas

$$(7) \qquad R^r{}_{ijk,r} = R_{ij,k} - R_{ik,j}, \qquad R^r{}_{j,r} = \tfrac{1}{2} R_{,j},$$

we obtain

$$R_{ij,k} - R_{ik,j} = \frac{1}{2(n-1)} (R_{,k} g_{ij} - R_{,j} g_{ik}),$$

or, by a covariant differentiation,

$$(8) \qquad R_{ij,kl} - R_{ik,jl} = \frac{1}{2(n-1)} (R_{,kl} g_{ij} - R_{,jl} g_{ik}).$$

It can be easily verified that (8) gives

$$R_{ik,lj} - R_{il,kj} = \frac{1}{2(n-1)} (R_{,lj} g_{ik} - R_{,kj} g_{il}),$$

$$R_{il,jk} - R_{ij,lk} = \frac{1}{2(n-1)} (R_{,jk} g_{il} - R_{,lk} g_{ij}),$$

which together with (8) again implies

$$(9) \qquad (R_{ij,kl} - R_{ij,lk}) + (R_{ik,lj} - R_{ik,jl}) + (R_{il,jk} - R_{il,kj}) = 0.$$

Applying now the Ricci identity to (9), we find

$$R_{rj} R^r{}_{ikl} + R_{ir} R^r{}_{jkl} + R_{rk} R^r{}_{ilj} + R_{ir} R^r{}_{klj} + R_{rl} R^r{}_{ijk} + R_{ir} R^r{}_{ljk} = 0.$$

However, the last equation leads immediately to (6) in view of

$$R_{ir} R^r{}_{jkl} + R_{ir} R^r{}_{klj} + R_{ir} R^r{}_{ljk} = 0,$$

which follows easily from Bianchi's identity. The lemma is proved.

From now on we assume that the considered conformally symmetric space is 2-Ricci-recurrent.

LEMMA 2. *The Ricci-tensor of a conformally symmetric 2-Ricci-recurrent space satisfies the condition*

$$(10) \qquad R_{ri} R^r{}_p = \frac{n}{2(n-1)} R R_{ip} - \frac{1}{4(n-1)} R^2 g_{ip}.$$

Proof. As an immediate consequence of (2), we have

$$(11) \qquad R_{,lm} = R a_{lm}.$$

This, together with (2), reduces (8) to the form

$$(12) \qquad \left(R_{ij} - \frac{1}{2(n-1)} R g_{ij}\right) a_{kl} = \left(R_{ik} - \frac{1}{2(n-1)} R g_{ik}\right) a_{jl}.$$

Transvecting now (12) with $R^j_{\ p}$ and using the relation

$$a_{rm}R^r_{\ j} = \tfrac{1}{2}Ra_{jm}, \tag{13}$$

which follows easily from (2), (7) and (11), we find

$$\frac{1}{2}R\left(R_{ik}-\frac{1}{2(n-1)}Rg_{ik}\right)a_{pl} = \left(R_{ri}R^r_{\ p}-\frac{1}{2(n-1)}RR_{ip}\right)a_{kl}. \tag{14}$$

But it follows from (12) that

$$\frac{1}{2}R\left(R_{ik}-\frac{1}{2(n-1)}Rg_{ik}\right)a_{pl} = \frac{1}{2}R\left(R_{ip}-\frac{1}{2(n-1)}Rg_{ip}\right)a_{kl}.$$

Substituting the last formula into (14), we get

$$\left(R_{ri}R^r_{\ p}-\frac{1}{2(n-1)}RR_{ip}\right)a_{kl} = \frac{1}{2}R\left(R_{ip}-\frac{1}{2(n-1)}Rg_{ip}\right)a_{kl},$$

which, evidently, is equivalent to (10).

In what follows we need the following result of Roy Chowdhury (cf. [7], Theorem 1 and equations (1.6), (2.3)):

LEMMA 3 (Roy Chowdhury). *The tensor of recurrence of a conformally symmetric 2-Ricci-recurrent space is symmetric and the Ricci-tensor satisfies the relations*

$$R_{rj}R^r_{\ ilm}+R_{ir}R^r_{\ jlm} = 0, \tag{15}$$

$$QR_{ij} = \frac{1}{2(n-1)}RQg_{ij}+\frac{n-2}{2(n-1)}Ra_{ij}, \tag{16}$$

where $Q = g^{ij}a_{ij}$.

LEMMA 4. *The curvature tensor of a conformally symmetric 2-Ricci-recurrent space satisfies the relation*

$$RR_{rp}R^r_{\ ikl} = RR_{rl}R^r_{\ kip}. \tag{17}$$

Proof. Transvecting (6) with $R^j_{\ p}$ and using identities $R_{rilj} = R_{ljri} = -R_{jlri} = R_{jlir}$, we obtain the equation

$$R_{rs}R^s_{\ p}R^r_{\ ikl}+R^r_{\ k}R_{sp}R^s_{\ lir}+R^r_{\ l}R_{sp}R^s_{\ kri} = 0,$$

which, because of (15), yields

$$R_{rs}R^s_{\ p}R^r_{\ ikl}-R^r_{\ k}R_{sl}R^s_{\ pir}-R^r_{\ l}R_{sk}R^s_{\ pri} = 0. \tag{18}$$

Transvecting now (18) with $R^l{}_m$ and applying (10), we find

$$\frac{n}{2(n-1)}R(R^l{}_m R^r{}_p R_{rikl} - R^r{}_k R_{sm} R^s{}_{pir} - R^r{}_m R_{sk} R^s{}_{pri}) -$$
$$-\frac{1}{4(n-1)}R^2(R^l{}_m R_{pikl} - R^r{}_k R_{mpir} - R_{sk} R^s{}_{pmi}) = 0,$$

whence

$$\frac{n}{2(n-1)}R(R^r{}_p R^s{}_m R_{riks} - R^r{}_k R_{sm} R^s{}_{pir} - R^r{}_m R_{sk} R^s{}_{pri}) -$$
$$-\frac{1}{4(n-1)}R^2(R_{rm} R^r{}_{kip} - R_{rk} R^r{}_{ipm} - R_{rk} R^r{}_{pmi}) = 0.$$

But the last equation, in virtue of (15), can be written in the form

$$\frac{n}{2(n-1)}R(R^r{}_p R^s{}_m R_{riks} - R^r{}_k R_{sm} R^s{}_{pir} - R^r{}_m R_{sk} R^s{}_{pri}) -$$
$$-\frac{1}{4(n-1)}R^2(R_{rm} R^r{}_{kip} + R_{ri} R^r{}_{kpm} + R_{rp} R^r{}_{kmi}) = 0,$$

which, because of (6), leads immediately to

(19) $$R(R^r{}_p R^s{}_m R_{riks} - R^r{}_k R_{sm} R^s{}_{pir} - R^r{}_m R_{sk} R^s{}_{pri}) = 0.$$

Comparing now (18) and (19), we easily obtain

(20) $$RR^r{}_p R^s{}_l R_{riks} = RR_{rs} R^s{}_p R^r{}_{ikl}.$$

Since $R_{riks} = R_{skir}$,

(21) $$RR^r{}_p R^s{}_l R_{riks} = RR^r{}_l R^s{}_p R_{rkis}.$$

But (20) can be written as

$$RR^r{}_l R^s{}_p R_{rkis} = RR_{rs} R^s{}_l R^r{}_{kip}.$$

The last equation, together with (20) and (21), gives

$$RR_{rs} R^s{}_l R^r{}_{kip} = RR_{rs} R^s{}_p R^r{}_{ikl},$$

whence, by substituting now (10), we get

$$\frac{n}{2(n-1)}R^2(R_{rp} R^r{}_{ikl} - R_{rl} R^r{}_{kip}) = \frac{1}{4(n-1)}R^3(R_{pikl} - R_{lkip}).$$

Since $R_{pikl} = R_{lkip}$, the last relation yields immediately (17).

3. Main results. Now we may proceed to the main results of this paper.

THEOREM 1. *The scalar curvature of a conformally symmetric 2-Ricci-recurrent space is zero at every point of the space.*

Proof. Transvecting (12) with $a^i_{\ p}$ and using (13), we obtain

$$Ra_{jp}a_{kl} = Ra_{kp}a_{jl}. \tag{22}$$

Since a_{ij} is symmetric and $a_{ij} \neq 0$ (at every point) by assumption, there obviously exists a real vector field v^j such that the condition $v^r v^s a_{rs} = e$ ($e = \pm 1$) holds in some neighbourhood U of an arbitrary fixed point P.

Therefore, transvecting (22) with $v^k v^l$ and putting $c_j = v^r a_{rj}$, we find in U

$$Ra_{jp} = eRc_j c_p. \tag{23}$$

On the other hand, it follows easily from (16) and (17) that

$$\frac{n-2}{2(n-1)} R^2 (a_{rp} R^r_{\ ikl} - a_{rl} R^r_{\ kip}) = \frac{1}{2(n-1)} R^2 Q (R_{lkip} - R_{pikl}),$$

whence, because of $R_{lkip} = R_{pikl}$, we get

$$R^2 a_{rp} R^r_{\ ikl} = R^2 a_{rl} R^r_{\ kip}.$$

But the last equation together with (23) gives in U

$$R^2 c_p c_r R^r_{\ ikl} = R^2 c_l c_r R^r_{\ kip}. \tag{24}$$

Since $c^s c_r R^r_{\ sip} = 0$, (24) implies $R^2 c_p c^s c_r R^r_{\ ils} = 0$, whence, using (24) again, we have at P

$$R^2 c^s c_s c_r R^r_{\ ikl} = 0. \tag{25}$$

It follows easily from (23) that the assumption $c^r c_r = 0$ leads to $RQ = 0$, which, in view of (16), implies $R = 0$. If, accordingly to (25), the equation $c_r R^r_{\ ikl} = 0$ holds, then, by contraction with g^{ik}, we find $c_r R^r_{\ l} = 0$. Transvecting now (10) with c^i and making use of the last relation, we obtain easily $Rc_p = 0$. Therefore, in all cases $R = 0$. Since P is arbitrary, our theorem is thus proved.

COROLLARY 1. *A conformally symmetric 2-Ricci-recurrent space with definite (positive or negative) metric does not exist.*

Indeed, as an immediate consequence of (10) and Theorem 1, we have $R_{ri} R^r_{\ p} = 0$, which, by contraction with g^{ip}, yields $R^{rs} R_{rs} = 0$, and in consequence $R_{ij} = 0$ — a contradiction with the definition of a 2-Ricci-recurrent space.

Remark. If a Riemannian n space ($n > 3$) satisfying (2) and (5) is analytic, then — as follows immediately from (23) and (25) — Theorem 1 and Corollary 1 both remain true under only assumption that a_{ij} and R_{ij} are not identically vanishing tensor fields.

THEOREM 2. *Conformally symmetric 2-Ricci-recurrent spaces are Ricci--recurrent spaces.*

Proof. As one can easily verify, in view of Theorem 1, equation (12) can be reduced to the form

$$a_{kl}R_{ij} = a_{jl}R_{ik}. \tag{26}$$

On the other hand, as an immediate consequence of

$$R_{ij,k}-R_{ik,j} = \frac{1}{2(n-1)}(R_{,k}g_{ij}-R_{,j}g_{ik})$$

(see proof of Lemma 1) and Theorem 1, we have

$$R_{ij,k} = R_{ik,j}. \tag{27}$$

Using (26), the symmetry of a_{ij} and the conditions $R = 0$ and $a_{ij} \neq 0 \neq R_{ij}$, one can verify (see [12], proof of Theorem 1) that there exists locally a real null vector field A_j such that

$$R_{ij} = eA_iA_j \quad \text{and} \quad a_{ij} = CA_iA_j, \tag{28}$$

where $C \neq 0$ and $e = \pm 1$.

Moreover, from Thompson's considerations (see [12], p. 509-510) it follows that if for a Riemannian space relations (2), (27) and (28) hold, then the space is necessarily a Ricci-recurrent one. This remark completes the proof of Theorem 2.

Combining Theorem 2 with Theorem 1 we have

COROLLARY 2. *Every conformally symmetric 2-Ricci-recurrent space is a Ricci-recurrent space with the necessarily vanishing scalar curvature.*

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