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REMARKS ON A MOMENT PROBLEM AND A PROBLEM OF PERFECTION OF POWER METHODS OF LIMITATION

BY

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We denote by R, R_+, R_- the intervals $(-\infty, +\infty)$, $\langle 0, +\infty)$, $(-\infty, 0)$, respectively. By a *measure* on an interval $P \subset R$ we shall understand a completely additive complex function defined on the family of Borel subsets of P.

Let $\varphi(t)$ be a positive continuous function on R such that

(1) $$\sup_{t\in R} \varphi(t)|t|^n < +\infty \quad \text{for } n = 0, 1, \ldots$$

We say that the function $\varphi(t)$ has *(M)-property* if for each number $r > 0$ and for each measure μ on R the condition

$$\int_R \varphi(t) t^n d\mu_t = O(r^n)$$

implies $\operatorname{supp}\mu \subset \langle -r, r\rangle$.

Analogously, if $\varphi(t)$ is a positive continuous function on R_+ such that

(2) $$\sup_{t\geqslant 0} \varphi(t) t^n < +\infty \quad \text{for } n = 0, 1, \ldots,$$

then we say that it has *(M^+)-property* if for each $r > 0$ and each measure μ on R_+ the condition

(3) $$\int_{R_+} \varphi(t) t^n d\mu_t = O(r^n)$$

implies $\operatorname{supp}\mu \subset \langle 0, r\rangle$.

C. Ryll-Nardzewski has proved in [5] that the Borel method of limitation is perfect. In fact, he has proved that the function e^{-t} has (M^+)-property (see also [2]). The aim of this note is a generalisation of the latter theorem.

The proof of C. Ryll-Nardzewski was based on the following theorem of Paley and Wiener:

(PW) *Let* $f(x)$ *be the Fourier transform of a measure* μ *on* R:

$$f(x) = \int_R e^{itx} d\mu_t, \qquad x \in R.$$

Then if $f(x)$ *can be extended to an entire function* $F(z)$ *of complex variable* $z = x+iy$, *satisfying*

$$(4) \qquad |F(z)| \leqslant Me^{r|z|},$$

then $\operatorname{supp}\mu \subset \langle -r, r\rangle$.

We shall now show that (PW)-theorem can be easily deduced from the (M^+)-property of the function e^{-t}.

Let the hypotheses of (PW)-theorem be satisfied.

We have

$$(5) \qquad F(x) = g(x)+h(x), \qquad x \in R,$$

where

$$g(z) = \int_{R_+} e^{itz} d\mu_t, \qquad y \geqslant 0,$$

$$h(z) = \int_{R_-} e^{itz} d\mu_t, \qquad y \leqslant 0.$$

It is easy to see that $g(z)$ (resp., $h(z)$) is the bounded continuous function for $y \geqslant 0$ (resp., for $y \leqslant 0$), holomorphic for $y > 0$ (resp., for $y < 0$). Thus we find, according to (5) and (4), that the functions $g(z)$, $h(z)$ can be extended to the entire functions $G(z)$, $H(z)$ satisfying

$$|G(z)| \leqslant Ne^{r|z|}, \qquad |H(z)| \leqslant Ne^{r|z|}.$$

The former inequality implies (according to Cauchy inequalities) $G^{(n)}(i) = O(\varrho^n)$ for each $\varrho > r$. On the other hand, we have

$$G^{(n)}(z) = \int_{R_+} e^{itz}(it)^n d\mu_t, \qquad y > 0,$$

and

$$G^{(n)}(i) = i^n \int_{R_+} e^{-t} t^n d\mu_t.$$

Thus

$$\int_{R_+} e^{-t} t^n d\mu_t = O(\varrho^n)$$

and, consequently,

$$\operatorname{supp}\mu \subset (-\infty, \varrho\rangle \quad \text{for every } \varrho > r,$$

i.e., $\operatorname{supp}\mu \subset (-\infty, r\rangle$. Similarly, after substitution $s = -t$ in the integral

$$f(-x) = \int_R e^{-itx} d\mu_t$$

we find that $\operatorname{supp}\mu \subset \langle -r, +\infty)$.

Now we shall prove the following

THEOREM 1. *Let $\varphi(t)$ be a positive continuous function on R. A sufficient condition for $\varphi(t)$ to have (M)-property is*

$$\sum_{n=1}^{\infty} \frac{1}{A_n^*} = +\infty, \tag{6}$$

where

$$A_n^* = \inf_{k\geqslant n} \sqrt[k]{A_k}, \qquad A_n = \sup_{t\in R} \varphi(t)|t|^n.$$

Proof. Let

$$\int_R \varphi(t) t^n d\mu_t = a_n r^n, \qquad |a_n| \leqslant M.$$

Let us write

$$f(x) = \int_R \varphi(t) e^{itx} d\mu_t \qquad (x \in R).$$

According to (6) there is $A_n < +\infty$ for $n = 0, 1, \ldots$ Consequently, (x) is infinitely many times differentiable in R and

$$f^{(n)}(x) = \int_R \varphi(t) e^{itx} (it)^n d\mu_t. \tag{7}$$

In particular,

$$f^{(n)}(0) = i^n \int_R \varphi(t) t^n d\mu_t = b_n r^n, \qquad |b_n| \leqslant M.$$

From (7) we get

$$|f^{(n)}(x)| \leqslant NA_n, \qquad x \in R,\ n = 0, 1, \ldots$$

Thus, according to Carleman's theorem (see [3]), the function $f(x)$ belongs to a quasi-analytic class.

Denoting

$$F(z) = \sum_{n=0}^{\infty} \frac{b_n r^n}{n!} z^n$$

we infer that

$$F^{(n)}(0) = b_n r^n = f^{(n)}(0), \qquad n = 0, 1, \ldots,$$

and

$$|F(z)| \leqslant M e^{r|z|}.$$

Thus we obtain $f(x) \equiv F(x)$ for $x \in R$, because $F(x)$ is an analytic function and $f(x)$ belongs to a quasi-analytic class.

But since

$$f(x) = \int_R e^{itx} d\nu_t, \qquad \text{where } d\nu_t = \varphi(t) d\mu_t,$$

we conclude (according to (PW)-theorem) that $\operatorname{supp}\nu \subset \langle -r, r\rangle$ and $\operatorname{supp}\mu \subset \langle -r, r\rangle$.

Remark. If the function $\varphi(t)$ fulfils the following additional condition

$$\sup_n \frac{1}{n} \sqrt[n]{A_n} < +\infty,$$

then the above proof becomes more elementary. Indeed, the function $f(x)$ is then analytic in R and the equality $f(x) \equiv F(x)$ is trivial. (Example: the function $\varphi(t) = e^{-|t|}$ has (M)-property, because $A_n = \sup_{t\geqslant 0} t^n e^{-t} = n^n e^{-n}$.)

THEOREM 2. *Let $\varphi(t)$ be a positive continuous function on R_+ satisfying (2). Then $\varphi(t)$ has (M^+)-property if and only if $\varphi(t^2)$ has (M)-property.* (Example: the function $e^{-\sqrt{t}}$ has (M^+)-property.)

Proof. 1° Suppose that $\varphi(t^2)$ has (M)-property and let (3) be satisfied. Integrating by substitution $t = s^2$ we get

$$O(r^n) = \int_{R_+} \varphi(t)\, t^n d\mu_t = \int_{R_+} \varphi(s^2) s^{2n} d\nu_s,$$

where $\nu(E) = \mu(E^2)$ for $E \subset R_+$, $E^2 = \{s^2\colon s \in E\}$. Now let us extend the measure ν to the "even" measure ω on R (i.e., $\omega(-E) = \omega(E)$ for $E \subset R$, where $-E = \{-s\colon s \in E\}$).

Then

$$\int_R \varphi(s^2) s^k d\omega_s = \begin{cases} 0 & \text{for } k = 2n-1, \\ 2\int_{R_+} \varphi(s^2) s^k d\nu_s & \text{for } k = 2n, \end{cases}$$

and, consequently,

$$\int_R \varphi(s^2) s^k d\omega_s = O\big((\sqrt{r})^k\big).$$

And the above condition implies $\operatorname{supp}\omega \subset \langle -\sqrt{r}, \sqrt{r}\rangle$, $\operatorname{supp}\nu \subset \langle 0, \sqrt{r}\rangle$, and $\operatorname{supp}\mu \subset \langle 0, r\rangle$.

2° Suppose now that $\varphi(t)$ has (M^+)-property and let

$$\int_R \varphi(t^2) t^n d\mu_t = O(r^n).$$

Let us denote by μ^e, μ^0 the "even" and "odd" part of the measure μ, i.e.,

$$\mu^e(E) = \tfrac{1}{2}\big(\mu(E) + \mu(-E)\big), \qquad \mu^0(E) = \tfrac{1}{2}\big(\mu(E) - \mu(-E)\big) \qquad \text{for } E \subset R.$$

Thus $\mu = \mu^e + \mu^0$ and we immediately obtain

$$\int_{R_+} \varphi(t^2) t^{2k} d\mu_t^e = O(r^{2k}), \qquad \int_{R_+} \varphi(t^2) t^{2k-1} d\mu_t^0 = O(r^{2k}),$$

whence, consequently,

$$\int_{R_+} \varphi(t^2) t^{2k} d\mu_t^e = \int_{R_+} s^k \varphi(s) d\nu_s = O\big((r^2)^k\big),$$

where $\mu^e(E) = \nu(E^2)$ for $E \subset R_+$. Thus $\operatorname{supp}\nu \subset \langle 0, r^2\rangle$ and $\operatorname{supp}\mu^e \subset \langle -r, r\rangle$.

Furthermore,

$$\int_{(r,+\infty)} \varphi(t^2) t^{2k-1} d\mu_t^0 = O(r^{2k}),$$

because

$$\int_{\langle 0,r\rangle} \varphi(t^2) t^{2k-1} d\mu_t^0 = O(r^{2k}).$$

But

$$\int_{(r,+\infty)} \varphi(t^2) t^{2k-1} d\mu_t^0 = \int_{(r,+\infty)} \varphi(t^2) t^{2k} d\omega_t,$$

where ω is the measure on $(r, +\infty)$ such that

$$d\omega_t = \frac{1}{t} d\mu_t^0.$$

After substitution $t^2 = s$ we get

$$\int_{(r^2,+\infty)} \varphi(s) s^k d\psi_s = O\big((r^2)^k\big),$$

(where $\omega(E) = \psi(E^2)$ for $E \subset (r, +\infty)$). Consequently, $\psi \equiv 0$, $\omega \equiv 0$ and, finally, $\operatorname{supp}\mu^0 \subset \langle -r, r\rangle$.

THEOREM 3. *Let $\varphi(t)$ be a positive continuous function on R, satisfying* (1). *Then a necessary condition for $\varphi(t)$ to have (M)-property is*

(8) $$\int_{-\infty}^{+\infty} \frac{\ln\varphi(t)}{1+t^2}\, dt = -\infty.$$

And if $\varphi(t)$ is of the form

(9) $$\varphi(t)^{-1} = \sum_{n=0}^{\infty} p_n t^{2n},$$

where $p_0 > 0$, $p_n \geqslant 0$ for $n = 1, 2, \ldots$, and the power series has infinite radius of convergence, then (8) *is also a sufficient condition.*

Proof. 1° If $\varphi(t)$ has (M)-property, then $\varphi(t)^{-1}$ is a weight function and, according to the theorem of Achiezer and Babienko [1], the condition (8) is valid.

2° If $\varphi(t)$ is of the form (9), then, according to the results of S. N. Bernstein (see [4]), the condition (8) is equivalent to the condition

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt[n]{A_n}} = +\infty, \quad \text{where } A_n = \sup_{t\in R} \varphi(t)|t|^n,$$

and the assertion of Theorem 3 follows from Theorem 1.

We shall now apply the two above theorems to the problem of perfection of power methods of limitation.

Let $p(t)$ be an entire function of the form

$$p(t) = \sum_{n=0}^{\infty} p_n t^n, \quad \text{where } p_n > 0 \text{ for } n = 0, 1, \ldots$$

Denote by $M(p)$ the method of limitation defined by the formula

$$\mathrm{LIM}(\xi) = \lim_{t\to+\infty} p(t)^{-1} \sum_{n=0}^{\infty} p_n \xi_n t^n,$$

where $\xi = (\xi_0, \xi_1, \ldots)$.

The author has shown in [2] that the method $M(p)$ is perfect if and only if the function $p(t)^{-1}$ has (M^+)-property. Thus according to Theorems 2 and 3 we get the following criterion:

THEOREM 4. *The power method $M(p)$ is perfect if and only if*

$$\int_{-\infty}^{+\infty} \frac{\ln p(t^2)}{1+t^2}\, dt = +\infty.$$

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