COLLOQUIUM MATHEMATICUM

VOL. LVII 1989 FASC. 1

SUBSYSTEMS OF THE SCHAUDER SYSTEM WHOSE ORTHONORMALIZATIONS ARE SCHAUDER BASES FOR $L^p[0, 1]$

BY

ROBERT E. ZINK (WEST LAFAYETTE, INDIANA)

1. If one performs the Gram–Schmidt maneuver on the system of Schauder functions, a standard Schauder basis for $C[0, 1]$, the orthonormal system that one obtains proves to be itself a Schauder basis for $C[0, 1]$ (Franklin [2]). It was shown by Szlenk [5], however, that this happy occurrence is merely a fluke, since one may perturb a Schauder system Φ in such a way that the perturbed system Φ_π is again a Schauder basis for $C[0, 1]$, while the Gram–Schmidt orthonormalization of the new system is not.

On the other hand, the Franklin system (the Gram–Schmidt (GS) orthonormalization of the Schauder system) is also a Schauder basis for each of the spaces $L^p[0, 1]$, $p \geqslant 1$, so that it is conceivable that $GS\Phi_\pi$ could be a basis for some of the L^p-spaces even though it fails to be a basis for $C[0, 1]$. Ancient results show that one of four situations must obtain: $GS\Phi_\pi$ is a Schauder basis for precisely those spaces $L^p[0, 1]$ with:

(1) $p \in [1, \infty]$, where $L^\infty[0, 1]$ is to be interpreted as $C[0, 1]$;

(2) $p \in (1, \infty)$;

(3) $p \in [\alpha/(\alpha-1), \alpha]$ for some $\alpha \geqslant 2$;

(4) $p \in (\alpha/(\alpha-1), \alpha)$ for some $\alpha > 2$.

In [6] Veselov has shown that each of these possibilities can be realized by appropriately choosing the parameters in Szlenk's example.

The present note is concerned with a different sort of disturbance of a Schauder system Φ. In this schema, one first deletes some of the elements of Φ and then applies the Gram–Schmidt process to the residual system Φ_ϱ. The orthonormal system $GS\Phi_\varrho$ (a generalized Franklin system) certainly will not be a Schauder basis for $C[0, 1]$, but if the L^p-closure of Φ_ϱ were all of $L^p[0, 1]$ for some values of p, it is conceivable that $GS\Phi_\varrho$ could constitute a Schauder basis for those spaces. Indeed, following a trail marked by Ciesielski [1] in his work on the Franklin functions, one finds that it is possible to delete a surprisingly extensive collection of elements from a Schauder system and still have a residual subsystem whose Gram–Schmidt orthonormalization is a Schauder basis for each space $L^p[0, 1]$, $1 \leqslant p < \infty$.

2. The Schauder systems herein considered are the usual collections of spike functions associated with a sequence

$$\{\pi_n = \{t_{nk}: 0 \leqslant k \leqslant 2^n\}\}_{n=0}^{\infty}$$

of subdivisions of [0, 1] satisfying:

$$t_{00} = 0, \quad t_{01} = 1;$$

$$t_{nk} < t_{n+1,2k+1} < t_{n,k+1} \quad \text{for all } n = 0, 1, \ldots \text{ and for all } k = 0, \ldots, 2^n - 1;$$

$$t_{n+1,2k} = t_{nk} \quad \text{for all } n = 0, 1, \ldots \text{ and for all } k = 0, \ldots, 2^n;$$

and

$$\lim_n \|\pi_n\| = 0.$$

The first two elements of the system are the constant function $\varphi_{00} = 1$ and the identity function φ_{01}. The remaining functions are defined in blocks of sizes 2^{n-1} ($n = 1, 2, \ldots$); $\varphi_{n,k-1}$, the k-th element of the n-th block, takes the value 1 at $t_{n,2k-1}$ and has for support the interval $(t_{n,2k-2}, t_{n,2k})$.

From a study of the almost-everywhere convergence of Schauder series, initiated by Goffman (see [3] and [7]), a simple criterion for the L^p-totality of a set of Schauder functions follows easily.

LEMMA 1. *Let $\Phi_Q = \{\varphi_1, \varphi_2, \ldots\}$ be an infinite subsystem of a Schauder system, and let $\{E_n\}_{n=1}^{\infty}$ be the sequence of supports of the elements of Φ_Q. In order that Φ_Q be total in each of the spaces $L^p[0, 1]$, $1 \leqslant p < \infty$, it is both necessary and sufficient that*

$$\mu(\limsup_n E_n) = 1. \tag{1}$$

(Indeed, should (1) fail to hold, Φ_Q would fail to be total in each of the L^p-spaces.)

THEOREM 2 (Ciesielski). *If Φ is a Schauder system, if $\{K_n\}_{n=1}^{\infty}$ is the sequence of Dirichlet kernels associated with the corresponding Franklin system, and if the sequence*

$$\{S_n: L^1[0, 1] \to C[0, 1]\}_{n=1}^{\infty}$$

is defined by

$$S_n f = \int_0^1 K_n(\cdot, t) f(t)\, dt,$$

then $\|S_n\|_p \leqslant 3$ for every $p \in [1, \infty]$.

The proof proceeds along the following lines. First, one observes that each linear combination

$$c_{00}\varphi_{00}+c_{01}\varphi_{01}+\sum_{i=1}^{n}\sum_{j=0}^{2^{n-1}-1}c_{ij}\varphi_{ij}+\sum_{j=0}^{r}c_{n+1,j}\varphi_{n+1,j}$$

is a polygonal function φ,

$$\varphi(t)=\frac{\xi_i-\xi_{i-1}}{t_i-t_{i-1}}(t-t_{i-1})+\xi_{i-1},\qquad t_{i-1}\leqslant t\leqslant t_i,\ i=1,\ldots,N,$$

where $\pi=\{t_0<t_1<\ldots<t_N\}$ is obtained by adjoining to π_n the elements $t_{n+1,1},\ldots,t_{n+1,2r+1}$ of π_{n+1}. The collection $\mathscr{F}$ of all polygonal functions of this kind coincides with the set of all Franklin polynomials whose orders do not exceed N. Let f be an element of $C[0, 1]$, and let I: $\mathscr{F}\to\mathscr{R}$ be defined by

$$I(\varphi)=\|\varphi-f\|_2^2\quad\text{for all } \varphi\in\mathscr{F}.$$

On the one hand, the minimum value of I is attained when $\varphi=S_N f$. On the other hand, necessary conditions for the attainment of this minimum are

$$\frac{\partial}{\partial\xi_i}I(\varphi)=0,\qquad i=0,\ldots,N,$$

from which one obtains the inequalities

$$|2\xi_0+\xi_1|\leqslant 3\|f\|_\infty,$$

$$\left|\frac{\delta_k}{\delta_k+\delta_{k+1}}\xi_{k-1}+2\xi_k+\frac{\delta_{k+1}}{\delta_k+\delta_{k+1}}\xi_{k+1}\right|\leqslant 3\|f\|_\infty,$$

$$|\xi_{N-1}+2\xi_N|\leqslant 3\|f\|_\infty,$$

where $\delta_i=t_i-t_{i-1}$, $i=1,\ldots,N$. Thus

$$\max\{|\xi_i|:\ i=0,\ldots,n\}=\|S_N f\|_\infty\leqslant 3\|f\|_\infty.$$

From the last inequality it follows that

$$\int_0^1|K_n(x, t)|\,dt\leqslant 3$$

for all x in $[0, 1]$. Finally, according to a theorem of Orlicz [4], if the elements of an orthonormal system are bounded and if, for every n,

$$\Big\|\int_0^1|K_n(\cdot, t)|\,dt\Big\|_\infty\leqslant A,$$

where A is a constant independent of n, then

$$\|S_n f\|_p\leqslant A\|f\|_p\quad\text{for all } p\in[1, \infty).$$

Thus, in the case of the Franklin system, one has

$$\|S_n f\|_p \leqslant 3\|f\|_p \quad \text{for all } p \in [1, \infty].$$

Since the Schauder system is total in each space $L^p[0, 1]$, $1 \leqslant p \leqslant \infty$, the Franklin system constitutes a Schauder basis for each of these spaces.

3. The procedure described above cannot be followed precisely when one deals with a proper subsystem of a Schauder system, because, in such a case, not all of the ξ_i are independent variables. Nevertheless, there is a large class of such subsystems to which the methods of Ciesielski can be applied.

THEOREM 3. *Let Φ_ϱ be a subsystem of a Schauder system Φ. If Φ_ϱ satisfies both* (1) *and*

(2) *for every $\varphi \in \Phi_\varrho$ and every $\psi \in \Phi$, if* $\operatorname{supp}\psi \subset \operatorname{supp}\varphi$, *then $\psi \in \Phi_\varrho$,*

then, for every p in $[1, \infty)$, $GS\Phi_\varrho$ is a Schauder basis for $L^p[0, 1]$.

Proof. Let $GS\Phi_\varrho = \{\psi_1, \psi_2, \ldots\}$ and let

$$\mathscr{P}_n = \Big\{\sum_{k=1}^{n} c_k \psi_k \colon c_k \in \boldsymbol{R},\ k = 1, \ldots, n\Big\}.$$

Each element of $\mathscr{P}_n$ is a polygonal function; indeed, $\mathscr{P}_n$ coincides with the set of all polygonal functions whose vertices have abscissae belonging to a fixed set

$$\{t_i\colon i = 0, 1, \ldots, N\}, \quad 0 = t_0 < t_1 < \ldots < t_N = 1,$$

determined by the vertices of $\psi_1, \ldots, \psi_n$. By virtue of (2), the n elements of Φ_ϱ, from which the elements of $\mathscr{P}_n$ are ultimately constructed, fall into subsets $\Phi_1, \ldots, \Phi_m$ such that the supports of two elements taken from different sets are disjunct and, for each i, there is an element of Φ_i whose support contains the support of every other element of that set. Let the supports of these superior functions be $(t_{j_k}, t_{j_k+r_k})$, $k = 1, \ldots, m$, and suppose that the notation has been chosen so that $t_{j_k+r_k} \leqslant t_{j_{k+1}}$, $k = 1, \ldots, m-1$. Then an element φ of $\mathscr{P}_n$ has the following form:

$$\varphi(t) = \begin{cases} 0 & \text{if } t \in [0, t_{j_1}] \cup [t_{j_1+r_1}, t_{j_2}] \cup \ldots \cup [t_{j_m+r_m}, 1];\\ \dfrac{\xi_{j_k+r} - \xi_{j_k+r-1}}{t_{j_k+r} - t_{j_k+r-1}}(t - t_{j_k+r-1}) + \xi_{j_k+r-1} & \text{if } t \in (t_{j_k+r-1}, t_{j_k+r}),\end{cases}$$

$$r = 1, \ldots, r_k;\ k = 1, \ldots, m.$$

Here $\xi_{j_k} = \xi_{j_k+r_k} = 0$ for $k = 1, \ldots, m$, but the other ordinates ξ_i can be independently assigned. In order that φ be the best L^2-approximant of the continuous function f, one must have, as before,

$$\frac{\partial}{\partial \xi_i}\|f-\varphi\|_2^2 = 0$$

for each independent variable. From these equations one obtains the following m sets of inequalities:

$$\left|2\xi_{j_k+1}+\frac{\delta_2}{\delta_1+\delta_2}\xi_{j_k+2}\right| \leqslant 3\|f\|_\infty,$$

$$\left|\frac{\delta_r}{\delta_r+\delta_{r+1}}\xi_{j_k+r-1}+2\xi_{j_k+r}+\frac{\delta_{r+1}}{\delta_r+\delta_{r+1}}\xi_{j_k+r+1}\right| \leqslant 3\|f\|_\infty,$$

$$\left|\frac{\delta_{r_k-1}}{\delta_{r_k-1}+\delta_{r_k}}\xi_{j_k+r_k-2}+2\xi_{j_k+r_k-1}\right| \leqslant 3\|f\|_\infty,$$

where $\delta_r = t_{j_k+r}-t_{j_k+r-1}$, $r = 1, \ldots, r_k$.

This elementary modification having been made, one sees that the proof can be speedily concluded by following precisely the route laid out by Ciesielski. As before, one finds, from the preceding inequalities, that $\|S_n f\|_\infty \leqslant 3\|f\|_\infty$ for every $f \in C[0, 1]$. From the afore-mentioned theorem of Orlicz it follows that

$$\|S_n\|_p \leqslant 3, \quad p \in [1, \infty),\ n = 1, 2, \ldots;$$

and the boundedness of the sequences $\{\|S_n\|_p\}_{n=1}^\infty$ together with (1) imply that

$$\lim_n \|S_n g - g\|_p = 0 \quad \text{for every } g \in L^p[0, 1],\ p \in [1, \infty).$$

While the condition (1) is a necessary one for the conclusion of Theorem 3, this is certainly not the case for (2).

THEOREM 4. *If Φ is the (standard) Schauder system associated with the sequence of binary subdivisions*

$$\{\pi_n = \{k/2^n\colon k = 0, 1, \ldots, 2^n\}\}_{n=0}^\infty,$$

and if Φ_Q is the residual system obtained from Φ by deleting any one of its elements, then $\mathrm{GS}\Phi_Q$ is a Schauder basis for each space $L^p[0, 1]$, $p \in [1, \infty)$.

Proof. Because (1) is satisfied in each such case, one need demonstrate only the boundedness of the sequences $\{\|S_n\|_p\}_{n=1}^\infty$. Since this demonstration is easily effected if the deleted element is either φ_{00} or φ_{01}, one supposes that the missing element is some φ_{nk} with $n \geqslant 1$. As in the argument used to establish the preceding theorem, it proves to be useful to examine the necessary conditions for the minimization of $I\colon \mathscr{P}_m \to \mathbf{R}$. For sufficiently large m, the polygonal functions $\varphi \in \mathscr{P}_m$ (may) have vertices at t_*, the midpoint of

E, the support of the function deleted. The ordinate ξ_* of this vertex will not be independent of the other ξ_i; indeed, $\xi_* = \frac{1}{2}(\xi_l + \xi_r)$, where ξ_l and ξ_r are the ordinates associated with the left- and right-end points of E. Let the set of all abscissae of vertices of elements of $\mathscr{P}_m$ be

$$\pi = \{0 = t_0 < t_1 < \ldots < t_N = 1\},$$

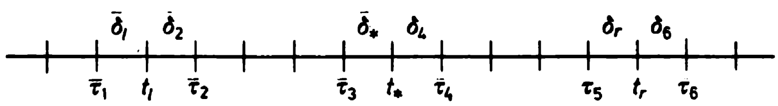
let $\varphi(t_i) = \xi_i$ for each i, and let

$$t_{j_1} = t_l, \quad t_{j_2} = t_*, \quad t_{j_3} = t_r.$$

A duplication of the earlier work leads to the inequalities

$$(3) \qquad 2|\xi_i| \leqslant 3\|f\|_{,} + \|\varphi\|_\infty, \quad i \notin \{j_1, j_2, j_3\}.$$

In order to make the notation somewhat more manageable, one relabels those elements of π that play rôles in the estimation of $|\xi_l|$ and $|\xi_r|$ in the following manner:



($\delta_l = 0$ if $t_l = 0$; $\delta_6 = 0$ if $t_r = 1$.)

Similarly, one relabels the corresponding ordinates:

$$\varphi(\tau_i) = y_i, \quad \varphi(t_l) = y_l, \quad \varphi(t_*) = y_*, \quad \varphi(t_r) = y_r.$$

The condition $\partial I/\partial y_l = 0$ yields

$$y_l \frac{\delta_l}{3} + y_1 \frac{\delta_l}{6} - \frac{1}{\delta_l} \int_{\tau_1}^{t_l} f(t)(t-\tau_1)\,dt$$

$$+ y_l \frac{\delta_2}{3} + y_2 \frac{\delta_2}{6} - \frac{1}{\delta_2} \int_{t_l}^{\tau_2} f(t)(\tau_2 - t)\,dt$$

$$+ \frac{1}{2}\left[y_* \frac{\delta_*}{3} + y_3 \frac{\delta_*}{6} \right] - \frac{1}{2\delta_*} \int_{\tau_3}^{t_*} f(t)(t-\tau_3)\,dt$$

$$+ \frac{1}{2}\left[y_* \frac{\delta_4}{3} + y_4 \frac{\delta_4}{6} \right] - \frac{\delta_1}{2\delta_4} \int_{t_*}^{\tau_4} f(t)(\tau_4 - t)\,dt = 0.$$

Consequently,

$$\left| y_l \frac{4\delta_l + 4\delta_2 + \delta_* + \delta_4}{2\delta_l + 2\delta_2 + \delta_* + \delta_4} + y_r \frac{\delta_* + \delta_4}{2\delta_l + 2\delta_2 + \delta_* + \delta_4} + \frac{2y_1\delta_l + 2y_2\delta_2 + y_3\delta_* + y_4\delta_4}{2\delta_l + 2\delta_2 + \delta_* + \delta_4} \right| \leqslant 3\|f\|_\infty.$$

Similarly, from $\partial I/\partial y_r = 0$ one obtains

$$\left| y_r \frac{4\delta_r + 4\delta_6 + \delta_* + \delta_4}{2\delta_r + 2\delta_6 + \delta_* + \delta_4} + y_l \frac{\delta_* + \delta_4}{2\delta_r + 2\delta_6 + \delta_* + \delta_4} + \frac{2y_5\,\delta_r + 2y_6\,\delta_6 + y_3\,\delta_* + y_4\,\delta_4}{2\delta_r + 2\delta_6 + \delta_* + \delta_4} \right| \leqslant 3\|f\|_\infty .$$

Taking account of the estimates (3), one obtains

$$|y_l| \frac{4\delta_l + 4\delta_2 + \delta_* + \delta_4}{2\delta_l + 2\delta_2 + \delta_* + \delta_4} \leqslant \frac{9}{2}\|f\|_\infty + \frac{1}{2}\|\varphi\|_\infty + |y_r| \frac{\delta_* + \delta_4}{2\delta_l + 2\delta_2 + \delta_* + \delta_4}$$

and

$$|y_r| \frac{4\delta_r + 4\delta_6 + \delta_* + \delta_4}{2\delta_r + 2\delta_6 + \delta_* + \delta_4} \leqslant \frac{9}{2}\|f\|_\infty + \frac{1}{2}\|\varphi\|_\infty + |y_l| \frac{\delta_* + \delta_4}{2\delta_r + 2\delta_6 + \delta_* + \delta_4}.$$

According as π coincides with one of the π_n or not, either the lengths of the intervals (t_{i-1}, t_i) are all the same or there is some i_0 such that $t_i - t_{i-1} = \delta$ for all $i \leqslant i_0$, and $\delta_i = 2\delta$ for all $i > i_0$. Suppose, first, that $t_l \neq 0$ and $t_r \neq 1$. Upon calculating the coefficients of $|y_l|$ and $|y_r|$ in each of the six possible cases:

(i) $\delta_l = \delta_2 = \delta_* = \delta_4 = \delta_r = \delta_6$,
(ii) $\delta_l = \delta,\ \delta_2 = \delta_* = \delta_4 = \delta_r = \delta_6 = 2\delta$,
(iii) $\delta_l = \delta_2 = \delta,\ \delta_* = \delta_4 = \delta_r = \delta_6 = 2\delta$,
(iv) $\delta_l = \delta_2 = \delta_* = \delta,\ \delta_4 = \delta_r = \delta_6 = 2\delta$,
(v) $\delta_l = \delta_2 = \delta_* = \delta_4 = \delta,\ \delta_r = \delta_6 = 2\delta$,
(vi) $\delta_l = \delta_2 = \delta_* = \delta_4 = \delta_r = \delta,\ \delta_6 = 2\delta$,

one obtains

$$\left\{\begin{matrix} \frac{5}{3} \\ \frac{8}{5} \\ \frac{3}{2} \\ \frac{11}{7} \\ \frac{5}{3} \\ \frac{5}{3} \end{matrix}\right\} |y_l| \leqslant \tfrac{9}{2}\|f\|_\infty + \tfrac{1}{2}\|\varphi\|_\infty + \left\{\begin{matrix} \frac{1}{3} \\ \frac{2}{5} \\ \frac{1}{2} \\ \frac{3}{7} \\ \frac{1}{3} \\ \frac{1}{3} \end{matrix}\right\} |y_r|$$

and

$$\left\{\begin{matrix} \frac{5}{3} \\ \frac{5}{3} \\ \frac{5}{3} \\ \frac{19}{11} \\ \frac{9}{5} \\ \frac{7}{4} \end{matrix}\right\} |y_r| \leqslant \tfrac{9}{2}\|f\|_\infty + \tfrac{1}{2}\|\varphi\|_\infty + \left\{\begin{matrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \\ \frac{3}{11} \\ \frac{1}{5} \\ \frac{1}{4} \end{matrix}\right\} |y_l|.$$

Thus, if $\|\varphi\|_\infty = |y_l|$, then

$$\|\varphi\|_\infty \leqslant 9\|f\|_\infty;$$

if $\|\varphi\|_\infty = |y_r|$, then

$$\|\varphi\|_\infty \leqslant \tfrac{27}{5}\|f\|_\infty.$$

Again, taking account of (3), it follows that $\|\varphi\|_\infty \leqslant 9\|f\|_\infty$.

If $t_l = 0$ or $t_r = 1$, a similar set of computations leads to the estimate

$$\|\varphi\|_\infty \leqslant 27\|f\|_\infty,$$

and the desideratum is attained.

4. It seems odd that so much effort was required in order to establish Theorem 4. Odder still, the resolution of the slightly more general problem, in which one deletes an arbitrary finite number of functions, appears to be a task of a much higher degree of complexity. One senses that the procedure followed in the preceding work is, quite likely, the wrong one; there should be some more general, and probably simpler, approach to these matters.

For example, a most helpful general principle would be the following: If Ψ and Φ are denumerable systems of continuous functions, if $\Psi \supset \Phi$, and if GSΦ is a Schauder basis for some space $L^p[0, 1]$, then GSΨ is also a Schauder basis for that space. Theorem 4 would follow swiftly from this "principle", since, if φ were the deleted function, it would suffice to delete, in addition, all of those Schauder functions whose supports are supersets of the support of φ and then to apply Theorem 3. Unfortunately, this would-be principle is false, as one can learn from a brief examination of the aforementioned Schauder bases introduced by Szlenk. Each of these systems Ψ is obtained from a Schauder system Φ by taking

$$\psi_{n0} = \varphi_{n0} + \varepsilon_n \varphi_{n,2^{n-1}-1} \quad \text{for } n = 2, 3, \ldots$$

and

$$\psi_{nk} = \varphi_{nk} \quad \text{in every other case.}$$

By appropriately choosing the sequence of underlying subdivisions $\{\pi_n\}_{n=0}^{\infty}$ and the sequence of perturbation constants $\{\varepsilon_n\}_{n=2}^{\infty}$, one can obtain a Szlenk system Ψ for which GSΨ is a Schauder basis only for $L^2[0, 1]$. On the other hand, the system Ψ obtained by deleting from Ψ all of the perturbed functions ψ_{n0} $(n = 2, 3, \ldots)$ coincides with a Schauder system Φ_ϱ to which Theorem 3 may be applied. Thus, $\Psi \supset \Psi_\varrho$, and GS$\Psi_\varrho$ is a Schauder basis for each space $L^p[0, 1]$, $1 \leqslant p < \infty$, while GSΨ is a basis only for $L^2[0, 1]$. General principles, alas, seem to be in somewhat short supply. Nevertheless, there must be a better mode of attack on the problem. Surely it must be true that Theorem 4 can be extended by permitting any finite number of deletions to be made. Indeed, one believes that Theorem 3 should continue to hold if one were to delete the hypothesis (2).

One final bit of strangeness is perhaps worthy of mention. The zero sets of the elements of a generalized Franklin system may have a very rich intersection. Indeed, one may arrange matters so that this intersection is a preassigned, nowhere dense, perfect null set.

Consider, e.g., the Cantor set C. Here one begins with a sequence of subdivisions of $[0, 1]$ in which the ternary fractions are featured. The first four of these are:

$$\{0, 1\}, \quad \{0, \tfrac{1}{3}, 1\}, \quad \{0, \tfrac{1}{9}, \tfrac{1}{3}, \tfrac{2}{3}, 1\},$$
$$\{0, \tfrac{1}{27}, \tfrac{1}{9}, \tfrac{2}{9}, \tfrac{1}{3}, \tfrac{1}{2}, \tfrac{2}{3}, \tfrac{7}{9}, 1\},$$

and the pattern is continued inductively. The new elements of π_{n+1} are given by

$$t_{n+1,2k+1} = \begin{cases} \frac{1}{2}(t_{nk}+t_{n,k+1}) & \text{if either } (t_{nk}+t_{n,k+1}) \text{ or } \left(t_{nk}, \frac{1}{2}(t_{nk}+t_{n,k+1})\right) \\ & \text{is a subset of } [0, 1]\setminus C, \\ \frac{2}{3}t_{nk}+\frac{1}{3}t_{n,k+1} & \text{in the contrary case.} \end{cases}$$

From the Schauder system based on $\{\pi_n\}_{n=0}^{\infty}$ one deletes those functions whose supports are not contained in the complement of C. The residual system satisfies the conditions of Theorem 3, and thus $GS\Phi_\varrho$ is a basis for $L^p[0, 1]$, $1 \leqslant p < \infty$, yet each of these generalized Franklin functions vanishes on the Cantor set.

REFERENCES

[1] Z. Ciesielski, *Properties of the orthonormal Franklin system*, Studia Math. 23 (1963), pp. 141–157.

[2] Ph. Franklin, *A set of continuous orthogonal functions*, Math. Ann. 100 (1928), pp. 522–529.

[3] C. Goffman, *Remark on a problem of Lusin*, Acta Math. 111 (1964), pp. 63–72.

[4] W. Orlicz, *Beiträge zur Theorie der Orthogonalentwicklungen*, Studia Math. 1 (1929), pp. 1–39.

[5] W. Szlenk, *Une remarque sur l'orthogonalisation des bases de Schauder dans l'espace C*, Colloq. Math. 15 (1966), pp. 297–301.

[6] V. M. Veselov, *On properties of bases after their orthogonalization* (in Russian), Izv. Akad. Nauk SSSR 34 (1970), pp. 1416–1436.

[7] R. E. Zink, *On a theorem of Goffman concerning Schauder series*, Proc. Amer. Math. Soc. 21 (1969), pp. 523–529.

DEPARTMENT OF MATHEMATICS
PURDUE UNIVERSITY
WEST LAFAYETTE, INDIANA 47907

Reçu par la Rédaction le 15.7.1983