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ON COUNTABLY COMPACT REDUCED PRODUCTS, III

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0. Introduction. The present paper contains a characterization of the family $\boldsymbol{R}$ of all ideals $\mathscr{I}$ of subsets of a set I having the following property: for every family $\mathscr{A} = \langle \mathfrak{A}_i : i \in I \rangle$ of similar relational structures (or finite structures) the direct product of $\mathscr{A}$ reduced by $\mathscr{I}$ is countably compact. Similar characterization for ω_1-universal reduced products are also given.

Investigation of compactness of reduced products was initiated by Keisler [7] who described, for every cardinal number $\varkappa$, the family of all maximal ideals $\mathscr{I}$ such that for every family $\mathscr{A}$ of relational structures the direct product of $\mathscr{A}$ reduced by $\mathscr{I}$ is $\varkappa^+$-compact. Also some other results on compactness of direct products reduced by maximal ideals were obtained by Keisler (see [9], where the incompactness of reduced products was studied).

The first result without assumption of the maximality of an ideal was obtained by Keisler [8]. He proved that the direct product of every family of Boolean algebras reduced by the Fréchet ideal (i.e. the ideal of all finite subsets of ω) is countably compact. Jónsson and Olin [6] generalized this result to products of arbitrary relational structures. Galvin [5] noticed that the Fréchet ideal in the theorem of Jónsson and Olin can be replaced by an arbitrary non-principal ideal having countable basis.

Conjecture of Jónsson and Olin that the case described by Galvin does not exhaust all ideals $\mathscr{I}$ from the family $\boldsymbol{R}$, even if we assume that the Boolean algebra of all subsets of I reduced by $\mathscr{I}$ is atomless (Boolean algebra of all subsets of an infinite set I reduced by a non-principal ideal $\mathscr{I}$, with countable basis is atomless), turned to be true and a characterization of the ideals from $\boldsymbol{R}$ with this additional assumption was obtained in a paper [10] by Ryll-Nardzewski and the author. Benda [1] announced generalizations of some results of [10].

The characterization of the family $\boldsymbol{R}$ presented here was independently obtained by Shelah [13] who was concerned with the more general case of $\varkappa^+$-compactness.

The paper is divided into 5 sections. In sections 1 and 2 we give the necessary background needed in sections 3 and 4. Section 3 contains some results concerning countably compact reduced products, section 4 deals with ω_1-universal reduced powers. In section 5 we prove some generalizations of a theorem of T. Skolem concerning elimination of quantifiers in the theory of atomless Boolean algebras.

Several results presented in this paper were announced in [11].

1. Notation and terminology. By $\mathfrak{A}$ (sometimes with subscripts) we denote relational structures (similarity type and first order language L are fixed), and the universe of a relational structure $\mathfrak{A}$ is denoted by A. We assume that every element of A has a name in L (excepting section 4). We assume that a set I is always infinite and every ideal is non-principal. **2** denotes the Boolean algebra $\langle\{\mathbf{0}, \mathbf{1}\}, \cup, \cap, -\rangle$. If $\mathscr{I}$ is an ideal of subsets of $\mathscr{I}$, then $\mathfrak{A}^I_{\mathscr{I}}$ denotes the power $\mathfrak{A}^I$ reduced by $\mathscr{I}$.

Let $\langle \mathfrak{A}_i: i \in I\rangle$ be a sequence of similar relational structures and $\mathscr{I}$ be an ideal of subsets of I. A subset X of the product $A = \mathsf{P}_{i\in I} A_i$ is called *definable in* $\mathfrak{A}_{\mathscr{I}} = \mathsf{P}_{i \in I} A_i/\mathscr{I}$ if for some formula φ of L we have

$$X = \{f \in A: \mathfrak{A}_{\mathscr{I}} \models \varphi[f/\mathscr{I}]\}.$$

By 2^{∞} we denote the set of all finite sequences of 0's and 1's.

A relational structure is said to be *countably compact* if the family of all sets definable in this structure is countably compact. For countable languages this notion is equivalent to the ω_1-saturatedness.

If L is a first order language and α a cardinal number, then $L(\alpha)$ denotes a language obtained from L by adding, for all $\beta < \alpha$, individual constants a_β. If $\mathfrak{A}$ is a relational structure and, for each $\beta < \alpha$, a_β is an element of A, then $(\mathfrak{A}, a_\beta)_{\beta<\alpha}$ denotes the structure obtained from $\mathfrak{A}$ by adding all a_β as distinguished elements.

A relational structure is called α^+-*universal* if, for every set Δ of sentences of $L(\alpha)$ which is consistent with $Th(\mathfrak{A})$, there is a model of $Th(\mathfrak{A})\cup\Delta$ of the form $(\mathfrak{A}, a_\beta)_{\beta<\alpha}$.

The theorem of S. Feferman and R. L. Vaught, which is very usefull in investigations of products of relational structures, is not applicable to reduced products. Dealing with reduced products one can use the Weinstein's extension of this theorem (see [4]) or one must change the notion of the reduced product treating equality as an equivalence relation.

Let $\langle \mathfrak{A}_i: i \in I\rangle$ be an indexed family of relational structures and $\mathfrak{A} = \mathsf{P}_{i\in I} A_i$. If θ is a formula of L with $n+1$ free variables and $f_k \in A$ for $k \leqslant n$, then we put

$$K[\mathfrak{A}, \theta](f_0, \ldots, f_n) = \{i \in I: \mathfrak{A}_i \models \theta[f_0(i), \ldots, f_n(i)]\}.$$

A sequence $\zeta = \langle \Phi, \vartheta_0, \ldots, \theta_m\rangle$ is called *acceptable* if Φ is a formula of the language of Boolean algebras and $\theta_0, \ldots, \theta_m$ are formulas of L.

An acceptable sequence ζ is called *partitioning* if the formulas $\bigvee\{\theta_i: i \leqslant m\}$ and $\neg(\theta_i \wedge \theta_j)$ for $i \neq j$ are logically valid ($\bigvee\{\theta_i: i \leqslant m\}$ denotes $\theta_0 \vee \ldots \vee \theta_m$).

Now we will state a version of the theorem of S. Feferman and R. L. Vaught which will be used in the sequel (cf. [3] and [4]).

THEOREM. *For every formula φ of L with $n+1$ free variables there exists a partitioning acceptable sequence $\zeta = \langle\Phi, \theta_0, \ldots, \theta_m\rangle$ such that, for every family $\langle\mathfrak{A}_i: i \in I\rangle$, every ideal $\mathscr{I}$ of subsets of I, and every sequence $\langle f_0, \ldots, f_n\rangle$ of elements of $\underset{i \in I}{\mathsf{P}} A_i$, we have $\underset{i \in I}{\mathsf{P}} \mathfrak{A}_i/\mathscr{I} \models \varphi[f]$ if and only if*

$$2^I_{\mathscr{I}} \models \Phi[K[A, \theta_0](f_0, \ldots, f_n)/\mathscr{I}, \ldots, K[\mathfrak{A}, \theta_m](f_0, \ldots, f_n)/\mathscr{I}].$$

2. Lemma. Before formulating the main result we will formulate some auxiliary statements. The author believes that the following refinement of a theorem of T. Skolem (see [14]) is well known (see Section 5):

PROPOSITION 1. *Let T be the theory of Boolean algebras. If Φ is a formula of the language of Boolean algebras with the variables $v_0, \ldots, v_{n-1}$, then*

$$T \vdash \Phi \leftrightarrow \bigwedge_{i<k} \bigvee_{j<m} \Phi_{i,j}(\tau_{i,j}),$$

where $\Phi_{i,j}$ is a formula with one variable $v_{i,j}$, and *$\tau_{i,j}$ is the term with the variables $v_0, \ldots, v_{n-1}$.*

DEFINITION 1. *If $A = \underset{i \in I}{\mathsf{P}} A_i$ and $\mathscr{I}$ is an ideal of subsets of I, then by $\mathscr{K}_{\mathscr{I}}(A)$ we denote the family*

$$\{\{f \in A: 2^I_{\mathscr{I}} \models \sigma[\{i: f(i) \in B_i\}/\mathscr{I}]\} : B_i \subseteq A_i$$

and σ is a formula of the language of Boolean algebras$\}$.

PROPOSITION 2. *For every sequence $\langle\mathfrak{A}_i: i \in I\rangle$ and every ideal $\mathscr{I}$ of subsets of a set I, the family of subsets of $A = \underset{i \in I}{\mathsf{P}} A_i$ definable in $\mathfrak{A}_{\mathscr{I}} = \underset{i \in I}{\mathsf{P}} \mathfrak{A}_i/\mathscr{I}$ is contained in the closure of $\mathscr{K}_{\mathscr{I}}(A)$ with respect to finite unions and intersections.*

. Proof. By Weinstein's refinement of the theorem of S. Feferman and R. L. Vaught, for any formula φ with one variable there exists a partitioning acceptable sequence $\zeta = \langle\Phi, \theta_0, \ldots, \theta_{n-1}\rangle$ such that for every $f \in A$

(1) $\mathfrak{A}_{\mathscr{I}} \models \varphi[f]$ *if and only if* $2^I_{\mathscr{I}} \models \Phi\big[K[\mathfrak{A}, \theta_0](f_0/\mathscr{I}, \ldots, K[\mathfrak{A}, \theta_m](f)/\mathscr{I}\big]$.

Applying Proposition 1 to Φ we get

$$2^I_{\mathscr{I}} \models \Phi \leftrightarrow \bigwedge_{i<k} \bigvee_{j<m} \Phi_{i,j}(\tau_{i,j}). \tag{2}$$

Since ζ is a partitioning sequence, we can assume without loss of generality that $\tau_{i,j}$ is the union of some v_i.

On the other hand,

$$\mathbf{2}^I_{\mathscr{I}} \models \psi(v_{k_1} \cup \ldots \cup v_{k_m})[\boldsymbol{K}[\mathfrak{A}, \theta_0](f)/\mathscr{I}, \ldots, \boldsymbol{K}[\mathfrak{A}, \theta_m](f)/\mathscr{I}]$$

if and only if

(3) $$\mathbf{2}^I_{\mathscr{I}} \models \psi[\{i \in I : \mathfrak{A}_i \models (\Theta_{k_0} \vee \ldots \vee \Theta_{k_m})[f(i)]\}/\mathscr{I}],$$

and, consequently, (3) holds if and only if

$$\mathbf{2}^I_{\mathscr{I}} \models \psi[\{i \in I : f(i) \in B_i\}],$$

where $B_i = \{a \in A_i : \mathfrak{A}_i \models (\Theta_{k_0} \vee \ldots \vee \Theta_{k_m})[a]\}$, which in view of (1) and (2) completes the proof.

LEMMA 1. *Let $\mathscr{I}$ be an ideal of subsets of a set I such that $\mathbf{2}^I_{\mathscr{I}}$ is countably compact. Then*

(a) *If I is the union of a countable subfamily of $\mathscr{I}$, then for every sequence $\langle \mathfrak{A}_i : i \in I \rangle$ of relational structures the family $\mathscr{K}_{\mathscr{I}}(A)$ is countably compact.*

(b) *For every sequence $\langle \mathfrak{A}_i : i \in I \rangle$ of finite relational structures the family $\mathscr{K}_{\mathscr{I}}(A)$ is countably compact.*

Proof. Let $\mathscr{C} = \langle C_j : j < \omega \rangle$ be a countable subfamily of $\mathscr{K}_{\mathscr{I}}(A)$ with the finite intersection property, where

(4) $$C_j = \{f \in A : \mathbf{2}^I_{\mathscr{I}} \models \sigma_j[\{i \in I : f(i) \in B_i^{(i)}\}/\mathscr{I}]\}.$$

For every sequence $\langle \varepsilon_0, \ldots, \varepsilon_n \rangle \in 2^\infty$ let

(5) $$b(\varepsilon_0, \ldots, \varepsilon_n) = \{i \in I : (B_i^{(0)})^{(\varepsilon_0)} \cap \ldots \cap (B_i^{(n)})^{(\varepsilon_n)} \neq 0\},$$

where $(B_i^{(j)})^{(1)} = B_i^{(j)}$ and $(B_i^{(j)})^{(0)} = A_i - B_i^{(j)}$.

Let

$$\Sigma = \{\sigma_j(v_j) : j < \omega\} \cup \{v_0^{(\varepsilon_0)} \cap \ldots \cap v_n^{(\varepsilon_n)} \subseteq b(\varepsilon_0, \ldots, \varepsilon_n) : \langle \varepsilon_0, \ldots, \varepsilon_n \rangle \in 2^\infty\}.$$

We will prove that Σ is finitely satisfiable in $\mathbf{2}^I_{\mathscr{I}}$. In fact, let

$$\Sigma_k = \{\sigma_j(v_j) : j \leqslant k\} \cup \{v_0^{(\varepsilon_0)} \cap \ldots \cap v_k^{(\varepsilon_k)} \subseteq b(\varepsilon_0, \ldots, \varepsilon_k) : \langle \varepsilon_0, \ldots, \varepsilon_k \rangle \in 2^\infty \text{ and } n \leqslant k\}.$$

Since $\mathscr{C}$ has the finite intersection property, there is a function f belonging to $\bigcap_{j \leqslant k} C_j$. For $j \leqslant k$ the set $a_j = \{i \in I : f(i) \in B_i^{(j)}\}/\mathscr{I}$ satisfies Σ_k in $\mathbf{2}^I_{\mathscr{I}}$. Since $\mathbf{2}^I_{\mathscr{I}}$ is countably compact, there is a sequence $\langle d_n : n < \omega \rangle$ of elements of $\mathbf{2}^I_{\mathscr{I}}$ satisfying Σ. Let $\langle D'_n : n < \omega \rangle$ be a sequence of subsets of I such that $D'_n/\mathscr{I} = d_n$. We shall define a sequence $\langle D_n : n < \omega \rangle$ of subsets of I such that

(6) $$D_n/\mathscr{I} = d_n$$

and

(7) $$D_0^{(\varepsilon_0)} \cap \ldots \cap D_n^{(\varepsilon_n)} \subseteq b(\varepsilon_0, \ldots, \varepsilon_n) \quad \text{for every } \langle \varepsilon_0, \ldots, \varepsilon_n \rangle \in 2^\infty.$$

If $D_0, \ldots, D_{n-1}$ are defined, put

$$D_n(\varepsilon_0, \ldots, \varepsilon_{n-1}) = (D_0^{(\varepsilon_0)} \cap \ldots \cap D_{n-1}^{(\varepsilon_{n-1})}) \cap \Big(\big(D'_n \cap b(\varepsilon_0, \ldots, \varepsilon_{n-1})\big) \cup \big(I - b(\varepsilon_0, \ldots, \varepsilon_{n-1}, 0)\big)\Big)$$

and

$$D_n = \bigcup \{D_n(\varepsilon_0, \ldots, \varepsilon_{n-1}) : \langle \varepsilon_0, \ldots, \varepsilon_{n-1}\rangle \in 2^{\infty}\}.$$

By an easy computation one verifies that D_n satisfy (6) and (7).

(a) Since I is the union of a countable subfamily of $\mathcal{I}$, there is a partition of I into sets E_n $(n < \omega)$ belonging to $\mathcal{I}$. We are going to define $f \in \cap \mathscr{C}$. If $i \in E_n$ and $\langle \varepsilon_0, \ldots, \varepsilon_n \rangle$ is a sequence such that $i \in D_0^{(\varepsilon_0)} \cap \ldots \cap D_n^{(\varepsilon_n)}$, then, by (5) and (7), $(B_i^{(0)})^{(\varepsilon_0)} \cap \ldots \cap (B_i^{(n)})^{(\varepsilon_n)} \neq 0$ and we put $f(i) = a$ for some $a \in (B_i^{(0)})^{(\varepsilon_0)} \cap \ldots \cap (B_i^{(n)})^{(\varepsilon_n)}$. Since $\{i : f(i) \in B_i^{(n)}\} \Delta D_n \subseteq E_n \in \mathcal{I}$ (Δ is the symmetric difference symbol), by (5) and (6) we have $f \in \bigcap \mathscr{C}$.

(b) Let $i \in I$. Since $\mathfrak{A}_i$ is finite, there is a natural number n such that if $j < \omega$, then $B_i^{(j)} = B_i^{(k)}$ for some $k < n$. Let $\langle \varepsilon_0, \ldots, \varepsilon_n \rangle$ be a sequence such that $i \in D_0^{(\varepsilon_0)} \cap \ldots \cap D_n^{(\varepsilon_n)}$. Put $f(i) = a$ for some a belonging to $(B_i^{(0)})^{(\varepsilon_0)} \cap \ldots \cap (B_i^{(n)})^{(\varepsilon_n)}$. It is easy to see that $\{i : f(i) \in B_i^{(n)}\} = D_n$, hence, by (5) and (6), we have $f \in \bigcap \mathscr{C}$.

3. Countably compact structures. Now we will formulate main results of this paper.

THEOREM 1. *If $2^I_{\mathcal{I}}$ is countably compact and I is the union of a countable subfamily of $\mathcal{I}$, then, for every sequence $\langle \mathfrak{A}_i : i \in I \rangle$ of similar relational structures, the reduced product $\mathfrak{A}_{\mathcal{I}} = \mathsf{P}_{i \in I} \mathfrak{A}_i / \mathcal{I}$ is countably compact.*

THEOREM 2. *If $2^I_{\mathcal{I}}$ is countably compact, then for every sequence $\langle \mathfrak{A}_i : i \in I \rangle$ of finite relational structures the reduced product $\mathfrak{A}_{\mathcal{I}} = \mathsf{P}_{i \in I} \mathfrak{A}_i / \mathcal{I}$ is countably compact.*

P r o o f o f T h e o r e m s 1 and 2. Obviously, it is enough to prove that the family of substes of $A = \mathsf{P}_{i \in I} A_i$ definable in $\mathfrak{A}_{\mathcal{I}}$ is countably compact, but this follows from Propositions 1 and 2 and from the known fact that the closure of a countably compact family of sets with respect to finite unions and intersections is countably compact.

Let us remark that the assumption that I is the union of countable subfamily of $\mathcal{I}$ is necessary. This is shown by the following example (cf. [10]). Consider a structure $\mathfrak{A}$ given by an infinite set A and a decreasing sequence of non-empty subsets B_n of A $(n < \omega)$ with the empty intersection. For any ideal $\mathcal{I}$ the sets $Q_n = \{f : \{i : f(i) \notin B_n\} \in \mathcal{I}\}$ form a decreasing sequence of non-void sets from $\mathscr{K}_{\mathcal{I}}(A^I)$. If $f \in \bigcap_{n<\omega} Q_n$, then the sets $E_n = \{i : f(i) \notin B_n\}$ belong to $\mathcal{I}$ and $\bigcup_{n<\omega} E_n = I$.

On the other hand, the countable compactness of $2^I_{\mathcal{I}}$ does not imply the existence of a partition of I into countably many sets from $\mathcal{I}$. An easy counterexample is provided by the algebra of all subsets of an uncountable set reduced by the ideal of all finite sets.

THEOREM 3. *For every complete theory T of Boolean algebras, there is an ideal $\mathcal{I}$ on ω such that $Th(\mathbf{2}^{\omega}_{\mathcal{I}}) = T$ and $\mathbf{2}^{\omega}_{\mathcal{I}}$ is countably compact.*

Proof. For every complete theory T of Boolean algebras there is an ideal $\mathcal{I}_0$ on ω such that $Th(\mathbf{2}^{\omega}_{\mathcal{I}_0}) = T$ (see [2]). Let $\mathcal{I}_1$ be a non-principal prime ideal on ω. We define an ideal $\mathcal{I}$ on $\omega \times \omega$ by putting $E \in \mathcal{I}$ if and only if

$$\{n \in \omega : \{i \in \omega : \langle n, i\rangle \in E\} \in \mathcal{I}_0\} \in \mathcal{I}_1.$$

Obviously $\mathbf{2}^{\omega\times\omega}_{\mathcal{I}} = (\mathbf{2}^{\omega}_{\mathcal{I}_0})^{\omega}_{\mathcal{I}_1}$, consequently, $Th(\mathbf{2}^{\omega\times\omega}_{\mathcal{I}}) = T$, and by a theorem of Keisler (see [7]) $\mathbf{2}^{\omega\times\omega}_{\mathcal{I}}$ is countably compact.

4. Universal structures. In this section we do not assume that every element of a structure has a name in L, but we assume that a language L is countable.

DEFINITION 2. *Let $\langle \mathfrak{A}_i : i \in I\rangle$ be a sequence of relational structures and $\mathcal{I}$ an ideal of subsets of I.*

(a) *If σ is a formula of the language of Boolean algebras with one free variable and ϑ a formula of L with $n+1$ free variables, then by $D_{\sigma,\vartheta}$ we denote the set*

$$\{\langle f_0, \ldots, f_n\rangle \in (\mathsf{P}_{i\in I} A_i)^{n+1} : \mathbf{2}^{I}_{\mathcal{I}} \models \sigma[K[\mathfrak{A}, \vartheta](f_0, \ldots, f_n)]/\mathcal{I}\}/\mathcal{I}.$$

(b) *By $\mathsf{P}^*_{i\in I}\mathfrak{A}_i/\mathcal{I}$ we denote the structure with the universe $\mathsf{P}_{i\in I} A_i/\mathcal{I}$ and with the relations $D_{\sigma,\vartheta}$. This structure is called a neat reduced product.*

The language of a neat reduced product of a sequence of structures with a language L is denoted by L^*.

(c) *Neat power of $\mathfrak{A}$ reduced by $\mathcal{I}$ is denoted by $\mathfrak{A}^*_{\mathcal{I}}$.*

Let us remark that a reduced product $\mathsf{P}_{i\in I}\mathfrak{A}_i/\mathcal{I}$ is a reduct of the neat reduced product $\mathsf{P}^*_{i\in I}\mathfrak{A}_i/\mathcal{I}$. This is an easy consequence of the theorem of S. Feferman and R. L. Vaught.

PROPOSITION 3. *Let $\langle \mathfrak{A}_i : i \in I\rangle$ be a sequence of relational structures with the same similarity type L, and $\mathcal{I}$ an ideal of subsets of I. Every formula of L^* is equivalent in the neat product $\mathsf{P}^*_{i\in I}\mathfrak{A}_i/\mathcal{I}$ to some open formula; more precisely, to some disjunction of conjunctions of atomic formulas.*

Proposition 3 is an immediate consequence of Proposition 1 and of the theorem of S. Feferman and R. L. Vaught [3] (cf. also [10], Proposition 1).

PROPOSITION 4. *Let T be a first order theory and let $\Delta = \{\bigvee\{\varphi_{n,j} : j \leqslant k_n\} : n < \omega\}$, where for $n < \omega$ and $j \leqslant k_n$ $\varphi_{n,j}$ is an atomic formula of* *a first order language L. If Δ is consistent with T, then there is a sequence $\langle m_n : n < \omega\rangle$ of natural numbers such that the set $\Delta' = \{\varphi_{n,m_n} : n < \omega\}$ is consistent with T.*

Proof. We define, by induction, a sequence $\langle m_n : n < \omega\rangle$ such that for $t < \omega$ the set

$$\{\varphi_{n,m_n} : n \leqslant t\} \cup \{\bigvee \{\varphi_{n,j} : j \leqslant k_n\} : t < n < \omega\}$$

is consistent with T. Let, for $n < t$, m_n be defined. Then at least one of the sets

$$\Delta_i = \{\varphi_{n,m_n} : n < t\} \cup \{\bigvee \{\varphi_{n,j} : j \leqslant k_n\} : t < n < \omega\} \cup \{\varphi_{t,m_i}\}$$

$(i \leqslant k_n)$ is consistent with T and we select m_t in a way such that Δ_{m_t} is consistent.

We say that a relational structure $\mathfrak{A}$ is *ω_1-universal with respect to atomic formulas* if for every set Δ of atomic formulas of $L(\omega)$ which is consistent with $Th(\mathfrak{A})$ there is a model of $Th(\mathfrak{A}) \cup \Delta$ of the form $(\mathfrak{A}, a_i)_{i<\omega}$.

LEMMA 2. *Let $\mathscr{I}$ be an ideal of subsets of a set I such that $\mathbf{2}^I_{\mathscr{I}}$ is ω_1-universal. Then*

(a) *If I is the union of a countable subfamily of $\mathscr{I}$, then for every relational structure $\mathfrak{A}$ the neat reduced power $\mathfrak{A}^*_{\mathscr{I}}$ is ω_1-universal with respect to atomic formulas.*

(b) *For every finite relational structure $\mathfrak{A}$ the neat reduced power $\mathfrak{A}^*_{\mathscr{I}}$ is ω_1-universal with respect to atomic formulas.*

Proof. Let Δ be a set of atomic formulas of $L^*(\omega)$. By adding apparent variables we can obtain $\Delta = \{D_{\sigma_n,\vartheta_n} : n < \omega\}$, where ϑ_n has $n+1$ free variables $x_0, \ldots, x_n$.

Let

$$\Sigma = \{\sigma_n(v_n) : n < \omega\} \cup \{v_0^{(\varepsilon_0)} \cap \ldots \cap v_n^{(\varepsilon_n)} = 0 : n < \omega,$$
$$\langle \varepsilon_0, \ldots, \varepsilon_n\rangle \in 2^\infty, \mathfrak{A} \models \neg \exists x_0 \ldots \exists x_n (\vartheta_0^{(\varepsilon_0)} \wedge \ldots \wedge \vartheta_n^{(\varepsilon_n)})\}.$$

We prove that $\Sigma \cup Th(\mathbf{2}^I_{\mathscr{I}})$ is consistent.

. Let

$$\Sigma_k = \{\sigma_n(v_n) : n \leqslant k\} \cup \{v_0^{(\varepsilon_0)} \cap \ldots \cap v_n^{(\varepsilon_n)} = 0 : n \leqslant k,$$
$$\langle \varepsilon_0, \ldots, \varepsilon_n\rangle \in 2^\infty, \mathfrak{A} \models \neg \exists x_0 \ldots \exists x_n (\vartheta_0^{(\varepsilon_0)} \wedge \ldots \wedge \vartheta_n^{(\varepsilon_n)})\}.$$

Since the set $Th(\mathfrak{A}^*_{\mathscr{I}}) \cup \Delta$ is consistent, there are functions $f_0, \ldots, f_k$ belonging to A^I such that $\mathfrak{A}_I \models D_{\sigma_n,\vartheta_n}[f_0, \ldots, f_n]$ for $n \leqslant k$. Let

$$B_n = \{i : \mathfrak{A} \models \vartheta_i[f_0(i), \ldots, f_n(i)]\}.$$

Of course, $\sigma_n(B_n/\mathscr{I})$, and if $\mathfrak{A} \models \neg \exists x_0 \ldots \exists x_n (\vartheta_0^{(\varepsilon_0)} \wedge \ldots \wedge \vartheta_n^{(\varepsilon_n)})$, then, for $i \in I$,

$$\mathfrak{A} \models \neg(\vartheta^{(\varepsilon_0)} \wedge \ldots \wedge \vartheta_n^{(\varepsilon_n)})[f_0(i), \ldots, f_n(i)].$$

Consequently, $B_0/\mathscr{I}, \ldots, B_k/\mathscr{I}$ satisfy Σ_k in $\mathbf{2}^I_{\mathscr{I}}$.

Since $\mathbf{2}^I_{\mathscr{I}}$ is ω_1-universal, there is a sequence $\langle c_n; n < \omega\rangle$ such that $(\mathbf{2}^I_{\mathscr{I}}, c_n)_{n<\omega}$ is a model of Σ. Let $\langle C_n : n < \omega\rangle$ be a sequence of subsets of I such that

$$C_n/\mathscr{I} = c_n \tag{8}$$

and

$$C_0^{(\varepsilon_0)} \cap \ldots \cap C_n^{(\varepsilon_n)} = 0 \quad \text{if} \quad c_0^{(\varepsilon_0)} \cap \ldots \cap c_n^{(\varepsilon_n)} = \mathbf{0}. \tag{9}$$

Now we will define a sequence $\langle g_n : n < \omega\rangle$ of functions such that $(\mathfrak{A}^*_{\mathscr{I}}, g_n)_{n<\omega}$ is a model of Δ.

(a) Let $\langle E_n : n < \omega\rangle$ be a partition of I into sets belonging to $\mathscr{I}$. If $i \in E_n$, then for some $\langle \varepsilon_0, \ldots, \varepsilon_n\rangle$ we have $i \in C_0^{(\varepsilon_0)} \cap \ldots \cap C_n^{(\varepsilon_n)}$.

Since $C_0^{(\varepsilon_0)} \cap \ldots \cap C_n^{(\varepsilon_n)}$ is non-empty and $(\mathbf{2}^I_{\mathscr{I}}, c_n)_{n<\omega}$ is a model of Σ, there is, by (8) and (9), a sequence $\langle a_k, k \leqslant n\rangle$ such that $\mathfrak{A} \models (\vartheta_0^{(\varepsilon_0)} \wedge \ldots \wedge \vartheta_n^{(\varepsilon_n)})[a_0, \ldots, a_n]$.

For $i \leqslant n$ we put $g_k(i) = a_k$ and for $i > n$ the element $g_k(i)$ is arbitrary. Similarly to the proof of Lemma 1 we prove that the sequence $\langle g_k : k < \omega\rangle$ has the desired properties.

(b) The construction of the sequence is left for the reader.

THEOREM 4. *Let $\mathscr{I}$ be an ideal on I. If the Boolean algebra $\mathbf{2}^I_{\mathscr{I}}$ is ω_1-universal and I is the union of a countable subfamily of $\mathscr{I}$, then for every relational structure $\mathfrak{A}$ the neat reduced power $\mathfrak{A}^*_{\mathscr{I}}$ is ω_1-universal.*

THEOREM 5. *If $\mathbf{2}^I_{\mathscr{I}}$ is ω_1-universal, then for every finite structure $\mathfrak{A}$ the neat reduced power $\mathfrak{A}^*_{\mathscr{I}}$ is ω_1-universal.*

Proof of Theorems 4 and 5. Let Δ be a set of formulas of $L^*(\omega)$ such that $Th(\mathfrak{A}^*_{\mathscr{I}}) \cup \Delta$ is consistent. By Proposition 3 every formula in Δ is equivalent in $\mathfrak{A}^*_{\mathscr{I}}$ to a conjunction of disjunctions of atomic formulas. Hence we can assume without loss of generality that Δ is a set of disjunctions of atomic formulas. By Proposition 4 every set Δ of disjunctions, which is consistent with $Th(\mathfrak{A}^*_{\mathscr{I}})$, can be replaced by a set of atomic formulas Δ' such that $\Delta' \cup Th(\mathfrak{A}^*_{\mathscr{I}})$ is consistent and such that for every formula $\varphi \in \Delta$ there is in Δ' a subformula ψ of φ. By Lemma 2 there is a sequence $\langle a_i : i < \omega\rangle$ such that $(\mathfrak{A}^*_{\mathscr{I}}, a_i)_{i<\omega}$ is a model of Δ'. By the definition of Δ' also the structure $(\mathfrak{A}^*_{\mathscr{I}}, a_i)_{i<\omega}$ is a model of Δ.

Since a reduced product is a reduct of a neat reduced product and a reduct of ω_1-universal structure is ω_1-universal we obtain the following corollaries:

COROLLARY 1. *If $\mathbf{2}^I_{\mathscr{I}}$ is ω_1-universal and I is the union of a countable subfamily of $\mathscr{I}$, then for every relational structure $\mathfrak{A}$ the reduced power $\mathfrak{A}^I_{\mathscr{I}}$ is ω_1-universal.*

COROLLARY 2. *If $\mathbf{2}^I_{\mathscr{I}}$ is ω_1-universal, then for every finite structure $\mathfrak{A}$ the reduced power $\mathfrak{A}^I_{\mathscr{I}}$ is ω_1-universal.*

Analogous theorems for products are false (see [9]). Also, answer to the question: Does $Th(\mathfrak{A}^I_{\mathscr{I}})$ have countable universal model, provided $Th(\mathbf{2}^I_{\mathscr{I}})$ has countable universal model? — is negative (see [11]).

5. Appendix. Elimination of quantifiers in the theory of Boolean algebras. Since some mathematicians do not share author's opinion that Proposition 1 is well known, we will give a sketch of the proof.

Let $\mathfrak{B}$ be a Boolean algebra and $\mathscr{I}(\mathfrak{B})$ the ideal of all elements which are joints of some atomic and atomless elements. Let $\mathfrak{B}_0 = \mathfrak{B}$ and $\mathfrak{B}_n = \mathfrak{B}_{n-1}/\mathscr{I}(\mathfrak{B}_{n-1})$ (see [2]). Let $a_i(x)$ be the formula of the language of Boolean algebras which says that $x/\mathscr{I}(\mathfrak{B}_i)$ is atomic and let $\beta_{i,j}(x)$ be the formula which says that $x/\mathscr{I}(\mathfrak{B}_i)$ has at least j atoms. The formulas a_i and $\beta_{i,j}$ are described in [2]. By L_1 we denote the language of Boolean algebras extended by symbols for a_i and $\beta_{i,j}$, and T_1 is the theory of Boolean algebras with defining axioms for a_i and $\beta_{i,j}$.

We will prove the following sentence stronger than Proposition 1:

5.1. *Every formula of Boolean algebras is equivalent in T_1 to some open formula of L_1.*

Let ψ be an open formula of L_1 with variables $v_0, \ldots, v_n$. It suffices to show that there exists an open formula φ of L_1 such that $T_1 \vdash \exists v_n \psi \leftrightarrow \varphi$. The proof of this fact will be devided into few stages.

5.2. *Every open formula of L_1 with variables $v_0, \ldots, v_n$ is equivalent in T_1 to a formula of the form*

$$(10) \qquad \bigvee \{ \bigwedge \{\varphi_j(\sigma_j \cap v_n^{(e_j)}) : k_i \leqslant j < k_{i+1}\} : i < l\},$$

where

$$(11) \qquad \sigma_j = v_0^{(\eta_0)} \cap \ldots \cap v_{n-1}^{(\eta_{n-1})}$$

and φ_j is an atomic formula or the negation of an atomic formula.

Proof. It suffices to prove that every atomic formula $\varphi(\tau)$ is equivalent to a formula of the form (10). By a standard method we replace the term τ by an equal term $\bigcup \{(\sigma_j \cap v_n^{(e_j)}) : j \leqslant m\}$, where σ_j is of the form (11). On the other hand, we have

$$T_1 \vdash a_i(\bigcup \{\tau_j : j \leqslant n\}) \leftrightarrow \bigwedge \{a_i(\tau_j) : j \leqslant n\}$$

and

$$T_1 \vdash \beta_{i,j}(\bigcup \{\tau_k : k \leqslant n\}) \leftrightarrow \bigvee \{ \bigwedge \{\beta_{i,j_k}(\tau_k) : k \leqslant n\} : \sum_{k \leqslant n} j_k = j\}.$$

5.3. *If $\varphi = \bigwedge \{\bigwedge \{\varphi_{i,j}(\sigma_j \cap v_n^{(e_i)}) : i \leqslant k_0\} : j \leqslant k_1\}$, then*

$$T_1 \vdash \exists v_n \varphi \leftrightarrow \bigwedge \{\exists v_n \bigwedge \{\varphi_{i,j}(\sigma_j \cap v_n^{(e_i)}) : i \leqslant k_0\} : j \leqslant k_1\}.$$

An easy proof of this fact is omitted.

5.4. *Let ψ be a formula*

(12) $$\bigwedge \{\varphi_i(\sigma \cap v_n^{(\varepsilon_i)}) : i \leqslant k\},$$

where σ_i and φ_i are as above. Then there exists an open formula ψ_0 such that

$$T_1 \vdash \exists v_n \psi \leftrightarrow \psi_0.$$

Proof. Let j_0 be the greatest natural number such that the formula a_{j_0} or the formula $\beta_{j_0,j}$, for some $j < \omega$, appears in ψ. We proceed by induction on j_0.

Let $\gamma_{j,k,0} = \beta_{j,k} \wedge \neg \beta_{j,k+1}$ and $\gamma_{j,k,1} = \beta_{j,k}$. If the formulas $\beta_{j_0,j_1}(\sigma \cap v_n^{(\varepsilon)})$ and $\neg \beta_{j_0,j_2}(\sigma \cap v_n^{(\varepsilon)})$ for $j_1 \leqslant j_2$ appear in the conjunction (12), then ψ is false. Otherwise ψ is equivalent in T_1 to disjunction of formulas of the form

(13) $$\gamma_{j_0,k_1,i_1}(\sigma_0 \cap v_n) \wedge a_{j_0}^{(\eta_1)}(\sigma \cap v_n) \wedge \psi_1 \wedge \gamma_{j_0,k_2,i_2}(\sigma_0 - v_n) \wedge a_{j_0}^{(\eta_2)}(\sigma - v_n) \wedge \psi_2,$$

where ψ_1 and ψ_2 are formulas of the form (12) such that 1° the formulas a_j and $\beta_{j,k}$ do not appear in $\psi_1 \wedge \psi_2$ for $j \geqslant j_0$ and $k < \omega$, 2° in atomic formulas in ψ_1 appears the term $\sigma \cap v_n$ only and in atomic formulas in ψ_2 appears the term $\sigma - v_n$ only.

Let ϑ be a formula of the form (13). We will show that $\exists v_n \vartheta$ is equivalent in T_1 to some open formula. Consider few cases.

Case 1. $k_i \geqslant 1$ or $\eta_i = 0$ and in ψ_i appears non-negated formula a_j or $\neg \beta_{j,k}$ for some $j < j_0$ and $i = 1$ or $i = 2$.

Then it is easy to check that ϑ is false in T_1.

Case 2. $k_i \geqslant 1$ or $\eta_i = 0$ for $i = 1$ or $i = 2$ and non-negated formula a_j and $\neg \beta_{j,k}$ does not appear in $\psi_1 \wedge \psi_2$ for any $j < j_0$.

Here

$$T_1 \vdash \vartheta \leftrightarrow \gamma_{j_0,k_1,i_1}(\sigma \cap v_n) \wedge a_{j_1}^{(\eta_1)}(\sigma \cap v_n) \wedge \gamma_{j_0,k_2,i_2}(\sigma - v_n) \wedge a_{j_2}^{(\eta_2)}(\sigma - v_n)$$

and one can verify that $\exists v_n \vartheta$ is equivalent in T_1 to some open formula (for instance, if $k_1 = i_1 = \eta_1 = 0$ and $k_2 = i_2 = \eta_2 = 1$, then $\exists v_n \vartheta \leftrightarrow \beta_{j_0+1,0}(\sigma) \wedge a_{j_0+1}(\sigma)$).

Case 3. $k_1 = k_2 = 0$, $\eta_1 = \eta_2 = 1$.

Here $T_1 \vdash \exists v_n \vartheta \leftrightarrow \exists v_n \psi_1 \wedge \psi_2$ and, by inductive hypothesis, there is an open formula φ such that $T_1 \vdash \exists v_n \vartheta \leftrightarrow \varphi$.

Combining 5.2 with 5.3 and 5.4 we obtain the proof of 5.1.

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