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ON A CLASS OF INDECOMPOSABLE CONTINUA

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In what follows all spaces are metric. The symbol B_0 stands for the simplest indecomposable continuum defined by B. Knaster (see [3], p. 204). A subset A of a space X *has the Baire property* if there exists an open subset U of X such that the set $(A \setminus U) \cup (U \setminus A)$ is of the first category in X. In 1932 Kuratowski proved that *if a set with the Baire property is the union of some composants of B_0, then either this set or its complement is of the first category in B_0.* The proof of this theorem, given by Kuratówski, cannot be applied to other continua. Hence the following question arises: is the above-given theorem true for any indecomposable continuum?. (**P 916**) This question seems to be difficult but partial results can be obtained.

In this note we prove that the Kuratowski theorem can be extended onto so-called simple indecomposable continua. It should be noted that the assumption "to have the Baire property" cannot be dropped from the hypothesis of the theorem. Actually, as was recently shown by Kuratowski, the collection of all composants of an indecomposable continuum X can be divided into two subcollections with the unions being second category subsets of X (not published). Our considerations are based on the following observation: there exists a point $p \in B_0$ (the end point of the accessible composant) and a sequence of open circular disks $U_1, U_2, \ldots$ (in the plane) with centre at p such that U_n's converge to p and, for each positive integer n, the decomposition of $B_0 \setminus U_n$ into its components is continuous. In our terminology this means that B_0 is *simple.*

1. Terminology and notation. Let X be a topological space. If A and B are subsets of X, then we say that the pair (A, B) *has property* (P) provided both A and B have this property. In this sense we say that (A, B) is open, closed, etc. If (C, D) is another pair in X, then we say that (A, B) *is in a relation R with* (C, D), written $(A, B)R(C, D)$, if both relations ARC and BRD hold. Hence it is clear what the following means: $(A, B) \subset (C, D)$, (A, B) is dense in (C, D), etc. A pair (A, B) is said to be *joined with respect to X* if every component of X which intersects A intersects also B. If, for every open and joined with respect to X pair (U, V) and for every open pair (U', V') which is dense in (U, V), there exists an open and joined with respect to X pair (U'', V'') such that $(\overline{U}'', \overline{V}'') \subset (U', V')$, then we say that X *has property* (α). We use the symbol $S(M, N)$ to denote the union of those components of M which intersect N. Using this notation we can equivalently say that (A, B) is joined if and only if $A \subset S(X, B)$.

Let D be a semi-continuous decomposition of X and let $f\colon X \to X/D$ be the quotient map. An open non-empty subset U of X is said to be a *jag* of D if $\operatorname{Int} f(U) = \emptyset$. It is clear that a continuous decomposition has no jag. Likewise, we say that a compactum X *has a jag* provided the (semi-continuous) decomposition of X into its components has a jag. An indecomposable continuum X is called *simple* provided there exists a closed non-empty set $F \subset X$ which does not meet all composants of X and satisfies the following condition: for every neighbourhood U of F, there exists an open set $V \supset F$ such that $\overline{V} \subset U$ and $X \setminus V$ has no jag.

2. Main result. We are going to prove the following

THEOREM. *Let X be a simple indecomposable continuum. If A is the union of some composants of X and if there exists a non-empty open subset U of X such that $(A \setminus U) \cup (U \setminus A)$ is of the first category, then U is dense in X.*

The following corollary generalizes the Kuratowski theorem [2]:

COROLLARY. *Let X be a simple indecomposable continuum. If a subset of X with the Baire property is the union of some composants of X, then either this set or its complement is of the first category in X.*

In fact, if A is such a subset of X, then $(U \setminus A) \cup (A \setminus U)$ is of the first category for some open subset U of X. If U is empty, then A is of the first category. Assume U is not empty. Then, by the Theorem, U is dense in X. To complete the proof it remains to note that $X \setminus A$ is of the first category since it can be written in the form

$$X \setminus A = [(X \setminus U) \setminus (A \setminus U)] \cup (U \setminus A),$$

where sets in brackets are of the first category.

Remark. In connection with this result let us note that in [3], Remark 3 of Section VI of § 48, the words "of a given continuum" should be replaced by "of the continuum".

The Theorem expresses a strong property of simple indecomposable continua, hence the following problems are of an interest:

PROBLEM 1. Does there exist an indecomposable continuum which is not simple? (**P 917**)

PROBLEM 2. Is the conclusion of the Theorem true for any indecomposable continuum? (**P 918**)

3. Auxiliary lemmas.

3.1. *If a compactum X has no jag, then it has property* (α).

Proof. Let (U, V) be an open pair joined with respect to X and let (U', V') be open and dense in (U, V). Let $f\colon X \to X/D$ be the quotient map, where D is the decomposition of X into its components. Finally, let $G = f^{-1}(\operatorname{Int} f(H))$, where H is a non-empty and open subset of U' such that

(1) $$\bar{H} \subset U'.$$

Then G is a non-empty open set such that $G = S(X, G \cap H)$. Moreover, the pair $(G \cap H, V)$ is joined with respect to X. It follows that $G \cap V'$ is non-empty since V' is dense in V. Let V'' be a non-empty open set such that

(2) $$\bar{V}'' \subset G \cap V'.$$

Set $U'' = H \cap f^{-1}(\operatorname{Int} f(V''))$. Since G is the union of some components of X, we obtain, by (2), $U'' \subset G \cap H$. This, by (1), implies

(3) $$\bar{U}'' \subset U'.$$

We claim that

(4) U'' is not empty.

To see this, let C be a component of $f^{-1}(\operatorname{Int} f(V''))$. Then, by (2), C is a subset of G. Therefore, C meets H and, by the definition of U'', we obtain (4).

By (2), (3) and (4) and by the construction of U'', we conclude that (U'', V'') is an open pair joined with respect to X and such that $(\bar{U}'', \bar{V}'') \subset (U, V)$, which completes the proof.

3.2. *Let X be a compactum with property* (α), *and let* (U_0, V_0) *be an open pair joined with respect to X. If (A, B) is a G_δ-pair dense in* (U_0, V_0), *then there exists a component of X which meets both A and B.*

Proof. By the hypothesis, there exist two sequences $U_0 \supset U_1 \supset \ldots$ and $V_0 \supset V_1 \supset \ldots$ of open subsets of X such that

$$A = \bigcap_n U_n \quad \text{and} \quad B = \bigcap_n V_n.$$

We can construct, by induction, a sequence $\{(G_n, H_n)\}$ of open pairs joined with respect to X, satisfying the following conditions:

(1) $$(\bar{G}_1, \bar{H}_1) \subset (U_1, V_1),$$

(2) $$(\bar{G}_n, \bar{H}_n) \subset (G_{n-1} \cap U_n, H_{n-1} \cap V_n) \quad \text{for } n = 2, 3, \ldots$$

Setting

(3) $$A' = \bigcap_n \bar{G}_n \quad \text{and} \quad B' = \bigcap_n \bar{H}_n,$$

we infer, by (1) and (2), that

(4) $$(A', B') \subset (A\cap G_n, B\cap H_n) \quad \text{for } n = 1, 2, \ldots$$

By (3), the sets A' and B' are non-empty. Let C be a component of X. which meets A'. Since A' is a subset of G_n, C meets H_n for each integer n. Thus (3) implies that C meets B'. By (4), the set C meets both A and B.

3.3. *Let (U, V) be a non-empty open pair in an indecomposable continuum X. Suppose that there exist a non-empty closed subset F of X and a strictly decreasing sequence $G_1 \supset \bar{G}_2 \supset G_2 \supset \bar{G}_3 \supset \ldots$ of open sets satisfying the following conditions:*

(i) $F = \bigcap_n G_n$,

(ii) *the set F does not meet all composants of X.*

Then there exist an index n_0 and an open pair $(U', V') \subset (U, V)$ which is joined with respect to $X\setminus G_{n_0}$ and such that $U'\cup V' \subset X\setminus G_{n_0}$.

Proof. We have
$$X = F\cup \bigcup_n (X\setminus G_n).$$

Since $F\cap V$ is a boundary subset of V, there exists an index n_1 such that

(1) $$V' = \operatorname{Int}(V\setminus G_{n_1}) \text{ is not empty}.$$

Let Q be a closed subset of X such that

(2) $$Q\subset V' \quad \text{and} \quad \operatorname{Int} Q \neq \emptyset.$$

Consider the closed sets $A_n = S(X\setminus G_n, Q)$, $n \geqslant n_1$. We show that

(3) $$U\subset C(F)\cup \bigcup_{n\geqslant n_1} A_n,$$

where $C(F)$ denotes the union of all those composants of X which meet F Let $x\in U\setminus C(F)$. By (2), the composant $C(x)$ meets Q. Hence there exists a continuum $B\subset C(x)$ joining x and Q. It follows that B is disjoint with F and, therefore, there exists an index $n_2 \geqslant n_1$ such that $B\subset X\setminus G_{n_2}$. Let B' be the component of $X\setminus G_{n_2}$ which contains B. By the definition of A_{n_2}, we obtain $x\in B\subset B'\subset A_{n_2}$ which proves (3).

According to a theorem of Cook (see [1], Theorem 1), the set $C(F)$ is a first category subset of X. Hence, by (3), there exists an index $n_0 \geqslant n_2$ such that

(4) $$U' = \operatorname{Int}(U\cap A_{n_0}) \text{ is not empty}.$$

It remains to show that (U', V') is a required pair. By (1) and (4), it is not empty and open. Moreover, $U'\subset A_{n_0}\subset X\setminus G_{n_0}$ and $V'\subset X\setminus G_{n_1} \subset X\setminus G_{n_0}$. Let C be a component of $X\setminus G_{n_0}$ which intersects U'. By (4), C meets Q. Hence, by (2), C meets V' which means that (U', V') is joined with respect to $X\setminus G_{n_0}$. This completes the proof.

3.4. *Let (U, V) be a non-empty open pair in a simple indecomposable continuum X. Then, for each pair $(M, N) \subset (U, V)$ which is a G_δ-pair dense in (U, V), there exists a composant of X which meets both M and N.*

Proof. By the hypothesis and by 3.1, there exist a non-empty closed set F and a strictly decreasing sequence $\{G_n\}$, $\bar{G}_{n+1} \subset G_n$, of open sets with the following properties:

(i) $F = \bigcap_n G_n$,

(ii) the set F does not meet all composants of X, and

(iii) for each integer n, the set $X \setminus G_n$ has property (α).

By 3.3, there exist an index n_0 and an open pair $(U', V') \subset (U, V)$ which is joined with respect to $X \setminus G_{n_0}$ and such that $U' \cup V' \subset X \setminus G_{n_0}$. Then $(M \cap U', N \cap V')$ is a G_δ-pair dense in (U', V'). Hence, by (iii) and 3.2, there exists a component D of $X \setminus G_{n_0}$ which meets both $M \cap U'$ and $N \cap V'$. Since D is a proper subcontinuum of X, the composant of X which contains D is a required one. This completes the proof.

4. Proof of the Theorem. Assume that the Theorem does not hold. Then there exist a non-empty open set $V \subset X \setminus U$ and nowhere dense subsets $M_1, M_2, \ldots$ and $N_1, N_2, \ldots$ such that

$$U \setminus A = \bigcup_n M_n \quad \text{and} \quad A \cap V = \bigcup_n N_n.$$

Setting

$$M = U \setminus \bigcup_n \bar{M}_n \quad \text{and} \quad N = V \setminus \bigcup_n \bar{N}_n,$$

we infer that

$$M \subset A, \quad A \cap N = \emptyset, \tag{1}$$

and (M, N) is a G_δ-pair dense in (U, V). According to 3.4, there exists a composant C which meets both M and N. But, by (1), we have $C \subset A$ (since A is the union of some composants), which implies that $\emptyset \neq C \cap N \subset A \cap N$, contrary to (1). This contradiction completes the proof.

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