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INTERPOLATION AND EXTENSION OF LIPSCHITZ-HÖLDER MAPS ON C_p SPACES

BY

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1. Introduction. A map F from a subset D of a metric space (X, d_1) to a metric space (Y, d_2) is said to be *Lipschitz continuous of order* a and belong to the class Lip (D, Y, a) provided

$$d_2(Fx_1, Fx_2) \leqslant [d_1(x_1, x_2)]^a \quad \text{for } x_1, x_2 \in D.$$

The statement that "$e(X, Y, a)$ holds" means that, for arbitrary $D \subset X$, every element of Lip(D, Y, a) can be extended to a map in Lip (X, Y, a).

Recently, interpolation theorems have been used in [1], [4] and [7] to prove extension theorems about Lipschitz-Hölder maps on function spaces. In this paper, a similar type interpolation theorem is proved which can then be applied to the C_p spaces defined below.

Let H be a Hilbert space and T a compact operator on H. Then $|T|$ is defined to be the unique positive square root of TT^* which is also compact. Now, let $\mu_1 \geqslant \mu_2 \geqslant \ldots \geqslant 0$ be the eigenvalues of $|T|$; then for $1 \leqslant p < \infty$ we define

$$\|T\|_p = \Big(\sum_{n=1}^{\infty} \mu_n^p\Big)^{1/p}.$$

The space C_p consists of all compact operators T such that $\|T\|_p$ is finite. The number $\|T\|_\infty$ will be the operator norm of T. For $p = 1$, the space is sometimes called the *trace class* or the *nuclear operators*. For details concerning the C_p spaces, see [2], [3] or [6].

2. Main results. Let $X_1, X_2, \ldots, X_n$ be Hilbert spaces and $P = (p_1, p_2, \ldots, p_n)$ an n-tuple of real numbers with $1 \leqslant p_k \leqslant \infty$. Define $\oplus C_{p_k}(X_k)$ to be the linear space of all vectors $T = (T_1, T_2, \ldots, T_n)$, $T_k \in C_{p_k}(X_k)$, with the usual coordinate-wise addition and scalar multiplication. In this space, introduce the norm

$$\|T\|_{p,r} = \Big\{\sum_{k=1}^{n} \|T_k\|_{p_k}^r \lambda_k\Big\}^{1/r},$$

where $1 \leqslant r < \infty$ and $\lambda = (\lambda_1, \ldots, \lambda_n\}$ is an n-tuple of positive weights.

In case $r = \infty$, write

$$\|T\|_{P,\infty} = \max_{1 \leqslant k \leqslant n} \|T_k\|_{p_k}.$$

Denote by $C_{P,r}(\lambda)$ the set of T such that $\|T\|_{P,r}$ is finite. If $P' = (p_1', p_2', \ldots, p_n')$, where $1/p_k + 1/p_k' = 1$, then for $S \in C_{P',r'}(\lambda)$

$$L(T) = \operatorname{tr} ST = \sum_{k=1}^{n} (\operatorname{tr} S_k T_k) \lambda_k$$

defines a bounded linear functional on $C_{p,r}$. Here tr denotes the trace of the operator. For $T \in C_{P,r}(\lambda)$,

$$\|T\|_{P,r} = \sup |\operatorname{tr} ST|, \tag{2.1}$$

where the supremum is taken over all finite-dimensional vectors belonging to $C_{P',r'}$ (cf. [2], p. 1098). A vector in $C_{P,r}$ is finite dimensional provided each component has finite-dimensional range.

Suppose $Y_1, Y_2, \ldots, Y_m$ is another collection of Hilbert spaces, $\eta = (\eta_1, \ldots, \eta_m)$ with $\eta_j \geqslant 0$ and $Q = (q_1, q_2, \ldots, q_m)$, $1 \leqslant q_i \leqslant \infty$. Define $C_{Q,s}(\eta)$ in a similar manner and consider linear maps taking finite-dimensional vectors of $C_{P,r}$ into the finite-dimensional vectors of $C_{Q,s}$.

THEOREM 2.1. *Suppose* $1 \leqslant P_i, Q_i \leqslant \infty, 1 \leqslant r_i, s_i \leqslant \infty$, $i = 1, 2$, *and* $1/P = (1-t)/P_1 + t/P_2$, $1/Q = (1-t)/Q_1 + t/Q_2$, $1/r = (1-t)/r_1 + t/r_2$, $1/s = (1-t)/s_1 + t/s_2$. *Let L be a bounded linear operator taking the finite-dimensional operators in C_{P_i,r_i} into the finite-dimensional operators in C_{Q_i,s_i}, $i = 1, 2$, with bounds M_1 and M_2, respectively. Then L takes $C_{P,r}$ into $C_{Q,s}$ and*

$$\|L(T)\|_{Q,s} \leqslant M_1^{1-t} M_2^t \|T\|_{P,r} \quad \textit{for } T \in C_{P,r}.$$

Proof. In explanation of the notation, $1/P = (1-t)/P_1 + t/P_2$ means $P = (p_1, \ldots, p_n)$, $P_1 = (p_{11}, p_{12}, \ldots, p_{1n})$, $P_2 = (p_{21}, \ldots, p_{2n})$ and $1/p_k = (1-t)/p_{1k} + t/p_{2k}$, $k = 1, 2, \ldots, n$.

It follows from (2.1) that all we have to show is that

$$|\operatorname{tr} L(T) S| \leqslant M_1^{1-t} M_2^t \tag{2.2}$$

for finite-dimensional operators T and S satisfying $\|T\|_{P,r} = 1$ and $\|S\|_{Q',s'} = 1$. Choose such a T and S and write $T_k = |T_k| U_k$, $S_k = |S_k| V_k$, where U_k and V_k are partial isometries obtained from the polar decomposition. If an operator H has the canonical representation

$$H = \sum_{i=1}^{\infty} \lambda_i (\cdot, \varphi_i) \varphi_i,$$

then by H^z we mean the operator

$$H^z = \sum_{i=1}^{\infty} \lambda_i^z (\cdot, \varphi_i) \varphi_i.$$

Let $\alpha_k(z) = (1-z)/p_{1k}+z/p_{2k}$, $1 \leqslant k \leqslant n$, $\beta_k(z) = (1-z)/q_{1k}+z/q_{2k}$, $\beta_k'(z) = (1-z)/q_{1k}'+z/q_{2k}'$, $\gamma(z) = (1-z)/r_1+z/r_2$, $\delta(z) = (1-z)/s_1+z/s_2$, and $\delta'(z) = (1-z)/s_1'+z/s_2'$. Define

$$F_k(z) = \begin{cases} \| \, |T_k| \, \|_{p_k}^{r\gamma(z)-p_k\alpha_k(z)} |T_k|^{p_k\alpha_k(z)} U_k, & p_k \neq \infty, \\ \| \, |T_k| \, \|_{p}^{r\gamma(z)-1} T_k, & p_k = \infty, \end{cases}$$

and

$$G_k(z) = \begin{cases} \| \, |S_k| \, \|^{s'\delta'(z)-q_k'\beta_k'} |S_k|^{q_k'(z)(1-\beta_k(z))} V_k, & q_k \neq 1, \\ \| \, |S_k| \, \|_{q_k'}^{s'\delta'(z)-1} S_k, & q_k = 1. \end{cases}$$

Let $F(z) = \big(F_1(z), F_2(z), \ldots, F_n(z)\big)$, $G(z) = \big(G_1(z), \ldots, G_m(z)\big)$; then, for $z = t$, $F(t) = T$ and $G(t) = S$.

Finally, define

$$\Phi(z) = \sum_{k=1}^{m} \operatorname{tr}\big[\big(LF(z)_k G_k(z)\big)\big] \eta_k.$$

Since $L(F)(z)_k$ and $G_k(z)$ are finite dimensional, it follows that $\Phi(z)$ is a holomorphic function on the strip $0 \leqslant \operatorname{Re} z \leqslant 1$.

$$\begin{aligned} \|F(1+iy)\|_{P_2,r_2}^{r_2} &= \sum_{k=1}^{n} \|F_k(1+iy)\|_{p_{2k}}^{r_2} \lambda_k \\ &= \sum_{k=1}^{n} \| \, |T_k| \, \|_{p_k}^{(r/r_2-p_k/p_{2k})r_2} \| \, |T_k|^{p_k/p_{2k}} \|_{p_{2k}}^{r_2} \lambda_k = 1. \end{aligned}$$

Similar computations show that $\|F(iy)\|_{p_1,r_1} = 1$ and $\|G(1+iy)\|_{Q_2',s_2'}^{s_2'} = \|G(iy)\|_{Q_1',s_1'}^{s_1'} = 1$.

These combine to imply

$$\begin{aligned} |\Phi(iy)| &= |\operatorname{tr}[L(F)(iy)G(iy)]| \\ &\leqslant \|LF(iy)\|_{Q_1,s_1} \|G(iy)\|_{Q_1',s_1'} \\ &\leqslant M_1 \|F(iy)\|_{P_1,r_2} \|G(iy)\|_{Q_1',s_1'} = M_1 \end{aligned}$$

and

$$|\Phi(1+iy)| \leqslant M_2 \|F(1+iy)\|_{P_2,r_2} \|G(1+iy)\|_{Q_2',s_2'} = M_2.$$

Thus applying the "three line theorem" we obtain (2.2).

This interpolation theorem can be applied to obtain various inequalities in C_p spaces. An example is

COROLLARY 2.1. *Let $2 \leqslant p < \infty$ and $p' \leqslant s \leqslant p$, where $1/p+1/p' = 1$. Then*

(2.3) $\quad \|T_1 - T_2\|_s^p + \|T_1+T_2\|_s^p \leqslant 2^{s-1}\{\|T_1\|_s^s + \|T_2\|_s^s\} \quad$ *for* $T_1, T_2 \in C_s$.

Proof. Note that when s is p or p', we get the usual Clarkson type inequalities obtained in [5].

Define the operator L on pairs of finite-dimensional operators by $L(T_1, T_2) = (T_1 - T_2, T_1 + T_2)$ and let $P_1 = Q_1 = (2, 2)$, $P_2 = Q_2 = (1, 1)$. The inequalities

$$\{\|T_1 - T_2\|_2^2 + \|T_1 + T_2\|_2^2\}^{1/2} \leqslant \sqrt{2}\{\|T_1\|_2^2 + \|T_2\|_2^2\}^{1/2} \tag{2.4}$$

and

$$\max\{\|T_1 - T_2\|_1, \|T_1 + T_2\|_1\} \leqslant \{\|T_1\|_1 + \|T_2\|_1\} \tag{2.5}$$

imply $L\colon C_{P_1,2} \to C_{Q_1,2}$ with $M_1 = \sqrt{2}$ and $L\colon C_{P_2,1} \to C_{Q_2,\infty}$ with $M_2 = 1$. Inequality (2.4) follows from properties of inner products. Choosing t so that $1/p' = (1-t)/2 + t = 1 + t/2$ and applying the interpolation theorem we see that $L\colon C_{P',p'} \to C_{P',p}$, where $P' = (p', p')$ and

$$\{\|T_1 - T_2\|_{p'}^p + \|T_1 + T_2\|_{p'}^p\}^{1/p} \leqslant 2^{1/p}\{\|T_1\|_{p'}^{p'} + \|T_2\|_{p'}^{p'}\}^{1/p'}. \tag{2.6}$$

Setting $P_1 = Q_1 = (2, 2)$, $P_2 = Q_2 = (\infty, \infty)$ and using the inequality

$$\max\{\|T_1 - T_2\|_\infty, \|T_1 + T_2\|_\infty\} \leqslant 2\max\{\|T_1\|_\infty, \|T_2\|_\infty\}, \tag{2.7}$$

we can apply the interpolation theorem to (2.4) and (2.7) with $1/p = (1-t)/2$ to see that $L\colon C_{P,p} \to C_{P,p}$, where $P = (p, p)$ and

$$\{\|T_1 - T_2\|_p^p + \|T_1 + T_2\|_p^p\}^{1/p} \leqslant 2^{1/p'}\{\|T_1\|_p^p + \|T_2\|_p^p\}^{1/p}. \tag{2.8}$$

Finally, apply the interpolation theorem to (2.6) and (2.8) with $1/s = (1-t)/p' + t/p$. Then $L\colon C_{S,s} \to C_{S,p}$ and (2.3) follows.

Theorem 2.1 can be used in exactly the same manner for the C_p spaces that the interpolation theorem is on L_p spaces in [4] to obtain the following

THEOREM 2.2. *Let H be any Hilbert space and $1 \leqslant p \leqslant \infty$. Then $e(C_p, H, a)$ holds provided $a \leqslant 1/2$ or $2a \leqslant p \leqslant 2a/(2a-1)$ whenever $1/2 \leqslant a \leqslant 1$.*

The reflexivity of C_p for $1 < p < \infty$ allows us to use the same techniques as in [7] to prove a similar theorem.

THEOREM 2.3. *Let $1 < p, q < \infty$, then $e(C_p, C_q, a)$ holds provided*

(i) $2a < p \leqslant 2$ and $p/(p-a) \leqslant q \leqslant p/a$

or

(ii) $2 \leqslant p < 2a/(2a-1)$ and $aq \leqslant p/(p-1)$.

The results could be extended to a larger class of spaces by using Young's functions to define the spaces as in [1].

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