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SUBBASE CHARACTERIZATION OF SPECIAL SPACES

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Many important topological spaces can be characterized by the fact that the spaces have a subbase with a special properties. For example, compact spaces (Alexander's subbase lemma), metrizable spaces [2], and products of orderable spaces (see [4], [5], [8], [9], [13], and [14]). Such characterizations will be called *subbase characterizations*.

The aim of this paper is to give several subbase characterizations of some special spaces: the products of special 0-dimensional compact spaces and metric separable prime spaces. Our technique allows us also to get in a simple way some known subbase characterizations mentioned above.

Following de Groot [7], a topological space X is called *supercompact* if it has a subbase for closed sets (*binary subbase*) such that each subfamily with empty intersection contains two disjoint elements.

We say that a subbase for closed sets of a space is *q-binary* if every finite subfamily with empty intersection contains two disjoint elements.

Spaces with a q-binary subbase have a natural product structure as is shown in Theorem 1. This theorem is the key for establishing subbase characterizations not necessarily of compact spaces.

A subbase $\mathcal{S}$ for the closed subsets of a space X is called *T_1-subbase* if for each $x \in X$ and $S \in \mathcal{S}$ such that $x \notin S$ there exists a $T \in \mathcal{S}$ with $x \in T$ and $S \cap T = \emptyset$.

A subbase $\mathcal{S}$ for the closed sets of a space X is said to be *normal* if for each $S, T \in \mathcal{S}$ with $S \cap T = \emptyset$ there exist S_1, $T_1 \in \mathcal{S}$ with $S \subset S_1$, $T \subset T_1$, $S \cap T_1 = \emptyset = T \cap S_1$, and $S_1 \cup T_1 = X$.

A subsystem $S \subset \mathcal{S}$ is called a *linked system* (ls) if every two its members meet. A linked system $M \subset \mathcal{S}$ is called *fixed* if $\bigcap M \neq \emptyset$, and *free* if $\bigcap M = \emptyset$. A *maximal linked system* (mls) in $\mathcal{S}$ is a linked system not properly contained in any other linked system. By the Kuratowski-Zorn lemma, every linked system is contained in at least one maximal linked system.

1. Let X be a T_1-space and $\mathscr{S}$ a subbase for closed sets. Let us assume that $\mathscr{S}$ is a q-binary and T_1-subbase.

The relation $\sim$ defined on $\mathscr{S}$ by $S \sim S'$ iff there exist $S_i \in \mathscr{S}, i = 0, \ldots, k$, such that $S_0 = S$, $S' = S_k$, and $S_{i-1} \cap S_i = \emptyset$ for each $i = 1, \ldots, k$ is an equivalence relation on $\mathscr{S}$.

Let us assume that $\mathscr{S}$ fulfils the following condition:

(F) If $\xi \subset \mathscr{S}$ has empty intersection, then there exists a $\xi' \subset \xi$ with empty intersection and such that each two elements from ξ' are in the relation $\sim$.

Let E be an equivalence class of the relation $\sim$. Then we put $X_E = \{M: M \subset E, M \text{ is an mls in } E \text{ and } \bigcap M \neq \emptyset\}$ endowed with the topology such that the sets $E(S) = \{M \in X_E: S \in M\}$, where S runs over the whole E, form a subbase for the closed sets.

Throughout the paper we fix the meanings of X, $\mathscr{S}$, $\sim$, E, and X_E as above.

LEMMA 1. *The family $\mathscr{S}_E = \{E(S): S \in E\}$ is a q-binary and T_1-subbase in X_E.*

Proof. Let $\xi \subset \mathscr{S}_E$ be a finite ls. Then $\xi' = \{S \in E: E(S) \in \xi\}$ is a finite ls in E. For $S, Z \in \xi'$ there is an mls M in E such that $M \in E(S) \cap \cap E(Z)$. Hence $S, Z \in M$. Since $E \subset \mathscr{S}$ and $\mathscr{S}$ is a q-binary subbase, ξ' is an ls with nonempty intersection.

To prove that $\mathscr{S}_E$ is a T_1-subbase let us take $e \in X_E$ and $E(S) \in \mathscr{S}_E$ with $e \notin E(S)$. Since e is an mls in E and $S \notin e$, there exists an $S' \in e$ such that $S' \cap S = \emptyset$. It follows that $E(S') \cap E(S) = \emptyset$. But, clearly, $e \in E(S')$.

For $p \in X$ we put $L_p = \{P \in E: p \in P\}$.

LEMMA 2. *For each $p \in X$ the family L_p is an mls in E (i.e., $L_p \in X_E$).*

Proof. Let $p \in X$. Since X is T_1, there exist two disjoint members in E. Hence $p \notin S$ for some $S \in E$. Since $\mathscr{S}$ is a T_1-subbase, there exists a $P \in \mathscr{S}$ such that $p \in P$ and $P \cap S = \emptyset$. Since $S \in E$ and $P \in E$, L_p is a nonempty ls in E. Assume that L_p is not an mls in E. Hence there exists a $Q \in E \setminus L_p$ such that $L_p \cup \{Q\}$ is an ls. Since $Q \notin L_p$, $p \notin Q$, and since $\mathscr{S}$ is a T_1-subbase, there exists a Q' in $\mathscr{S}$ with $p \in Q'$ and $Q' \cap Q = \emptyset$. Since $Q \in E$ and $Q' \cap Q = \emptyset$, we have $Q' \in E$ and $Q' \in L_p$; a contradiction with L_p being an ls.

LEMMA 3. *Cardinality of X_E is at least 2.*

Proof. The lemma follows immediately from the fact that X is a T_1-space and $\mathscr{S}$ is a T_1-subbase.

Assigning, by Lemma 2, to each $p \in X$ the mls L_p from X_E, we get a map $\varphi_E: X \to X_E$.

The map φ_E is onto. To see this let us take $M \in X_E$. Then M is an mls with nonempty intersection. Let $p \in \bigcap M$. Then L_p and M are mls with $p \in \bigcap M \cap \bigcap L_p$. Hence $L_p = M$.

The map φ_E is continuous. To see this it suffices to observe that $\varphi_E^{-1}(E(S)) = S$ for each $S \in E$.

Now, let $\mathcal{S}^{\sim}$ be the family of all equivalence classes of the relation $\sim$. Let Y denote the product of spaces X_E for each $E \in \mathcal{S}^{\sim}$ and let $\varphi\colon X \to Y$ be the diagonal of maps φ_E for each $E \in \mathcal{S}^{\sim}$ (i.e., φ is given by the formula $\varphi(p) = \{\varphi_E(p)\colon E \in \mathcal{S}^{\sim}\}$).

THEOREM 1. *The space X is homeomorphic to Y.*

Proof. The map φ, the diagonal of continuous maps, is continuous. Before proving that φ is onto observe that if $E, E' \in \mathcal{S}^{\sim}$, $E' \neq E$, $e \in X_E$, and $e' \in X_{E'}$, then $e \cup e'$ is an ls in $\mathcal{S}$. Now, if $y \in Y$ and $y = \{y_E\colon E \in \mathcal{S}^{\sim}\}$, then $\bigcup\{y_E\colon E \in \mathcal{S}^{\sim}\}$ is an ls in $\mathcal{S}$, and hence, by (F), there exists a $q \in \bigcap\{\bigcup\{y_E\colon E \in \mathcal{S}^{\sim}\}\}$. Clearly, for such a q, $\varphi_E(q) = y_E$ for each $E \in \mathcal{S}^{\sim}$. Thus $\varphi(q) = y$.

Let p and q be different points from X. Since X is a T_1-space and $\mathcal{S}$ is a T_1-subbase, there exist S and S' in $\mathcal{S}$ such that $p \in S$, $q \in S'$, and $S \cap S' = \emptyset$. Therefore, $S' \sim S$ and $S, S' \in E$ for some $E \in \mathcal{S}^{\sim}$. Hence $\varphi_E(p) \neq \varphi_E(q)$. Consequently, $\varphi(p) \neq \varphi(q)$.

The map φ is closed. To prove this it suffices to show only that the images of sets from $\mathcal{S}$ are closed, since φ is one-to-one and $\mathcal{S}$ is a closed subbase. Let $S \in \mathcal{S}$. Then $S \in E$ for some $E \in \mathcal{S}^{\sim}$. We shall show that

$$\varphi(S) = \varphi_E(S) \times \prod\{X_{E'}\colon E' \in \mathcal{S}^{\sim} \text{ and } E' \neq E\}.$$

Let us take $\varphi_{E'}(S)$, where $S \notin E'$. Suppose, on the contrary, that $\varphi_{E'}(S) \neq X_{E'}$. Let us take $M \in X_{E'} \setminus \varphi_{E'}(S)$. Hence, by (F), $S \cap \bigcap M \neq \emptyset$. Now, if we take a point $p \in S \cap \bigcap M$, then $L_p = M$. Hence $M \in \varphi_{E'}(S)$, a contradiction. Clearly, if $S \in E$, then $\varphi_E(S) = E(S)$. The proof is complete.

For any family A of subsets of a set X we will use $\vee A$ for the family of finite unions of elements of A and $\wedge A$ for the family of finite intersections of elements of A. The family $\wedge\vee A = \vee\wedge A$ is closed both under finite intersections and finite unions; it is called the *ring generated by A*.

A space X is called *regular supercompact* if X has a binary closed subbase $\mathcal{S}$ such that $\vee\wedge\mathcal{S}$ is a ring consisting of regularly closed sets.

Let X be a regular supercompact space and let $\mathcal{S}$ be a binary closed subbase such that $\vee\wedge\mathcal{S}$ is a ring consisting of regularly closed sets. It is easy to see that if D is a dense subset of X, then $\mathcal{S}|D = \{S \cap D\colon S \in \mathcal{S}\}$ is a q-binary subbase in D. Van Douwen [6] proved that each compact metric space is regular supercompact. Hence each separable metric space has a q-binary subbase.

A space X is said to be *prime* if it contains at least two points and there exists no Cartesian decomposition of X with at least two factors, each containing at least two points.

It is obvious that if X is the product of spaces X_α with a q-binary subbase $\mathscr{S}_\alpha$, $\alpha \in \Sigma$, then $\mathscr{S} = \{\pi_\alpha^{-1}(S)\colon\ \alpha \in \Sigma,\ S \in \mathscr{S}_\alpha\}$, where π_α's are natural projections, is a q-binary subbase in X, and if $\pi_\alpha^{-1}(S_1)$, $\pi_\beta^{-1}(S_2) \in \mathscr{S}$ are in the relation $\sim$ on $\mathscr{S}$, then $\alpha = \beta$, $S_1, S_2 \in \mathscr{S}_\alpha$, and S_1, S_2 are in the relation $\sim$ on $\mathscr{S}_\alpha$. Thus we have

THEOREM 2. *A separable metric space is prime if and only if each two members of an arbitrary q-binary subbase in the space are in the relation $\sim$.*

LEMMA 4. *Assume that X is a T_1-space and $\mathscr{S}$ a q-binary T_1-subbase in X which fulfils conditions* (F) *and*

(vI) *If S_1, S_2, S_3 are in $\mathscr{S}$ and $S_1 \cap S_2 = \emptyset = S_2 \cap S_3$, then $S_1 \subset S_3$ or $S_3 \subset S_1$, or $S_1 \cap S_3 = \emptyset$.*

Then for each two elements Z_1, Z_2 from the equivalence class E of the relation $\sim$ we have

(∗) *$Z_1 \cap Z_2 = \emptyset$ or $Z_1 \subset Z_2$, or $Z_2 \subset Z_1$, or $Z_1 \cup Z_2 = X$.*

Proof. Let Z_1, Z_2 be in E. Then there are $S_0, \ldots, S_k$ in $\mathscr{S}$ such that $Z_1 = S_0, Z_2 = S_k$, and $S_{i-1} \cap S_i = \emptyset$ for $i = 1, \ldots, k$. Now we prove condition (∗) by induction on k.

If $k = 1$, then condition (∗) holds.

Suppose that condition (∗) holds for all $r \leqslant k$. From the induction assumption it follows that the pairs Z_1 and S_{k-1}, Z_2 and S_1 fulfil (∗). Hence

(1) $Z_1 \cap S_{k-1} = \emptyset$ or (2) $Z_1 \subset S_{k-1}$, or (3) $S_{k-1} \subset Z_1$, or (4) $S_{k-1} \cup Z_1 = X$.

If (1) holds, then $Z_1 \cap S_{k-1} = \emptyset = S_{k-1} \cap Z_2$. Hence, by (vI), $Z_1 \cap Z_2 = \emptyset$ or $Z_1 \subset Z_2$, or $Z_2 \subset Z_1$.

If (2) is satisfied, then $Z_1 \subset S_{k-1}$ and $S_{k-1} \cap Z_2 = \emptyset$. Hence $Z_1 \cap Z_2 = \emptyset$.

If (3) holds, then we shall show that $Z_1 \cap Z_2 = \emptyset$ or $Z_2 \subset Z_1$, or $Z_1 \cup Z_2 = X$ (since it follows from (3) that $Z_1 \not\subset Z_2$).

Let us assume that $Z_2 \cup Z_1 \neq X$. Since $\mathscr{S}$ is a T_1-subbase, we can find a point $x \notin Z_1 \cup Z_2$ and sets $F_1, F_2 \in \mathscr{S}$ such that $x \in F_1 \cap F_2$ and $Z_2 \cap F_2 = \emptyset = Z_1 \cap F_1$. Since $S_{k-1} \subset Z_1$, we have $S_{k-1} \cap F_1 = \emptyset = Z_2 \cap S_{k-1}$. It follows from (vI) that $F_1 \cap Z_2 = \emptyset$ or $F_1 \subset Z_2$, or $Z_2 \subset F_1$.

If $Z_2 \cap F_1 = \emptyset$, then $Z_1 \subset Z_2$ or $Z_2 \subset Z_1$, or $Z_1 \cap Z_2 = \emptyset$, since $Z_1 \cap F_1 = \emptyset$.

The case $F_1 \subset Z_2$ is impossible as $x \in F_1$ and $x \notin Z_2$.

If $Z_2 \subset F_1$, then $Z_1 \cap Z_2 = \emptyset$ since $F_1 \cap Z_1 = \emptyset$.

If (4) holds, then $Z_2 \subset X \setminus S_{k-1} \subset Z_1$ since $S_{k-1} \cap Z_2 = \emptyset$.

Let X be a T_1-space and $\mathscr{S}$ a q-binary T_1-subbase which fulfils conditions (F) and

(I) If S_1, S_2, S_3 are in $\mathscr{S}$ and $S_1 \cap S_2 = \emptyset = S_2 \cap S_3$, then $S_1 \subset S_3$ or $S_3 \subset S_1$.

For each $S \in \mathscr{S}$ let us define a set

$$C(S) = \{Z \in \mathscr{S}: Z \subset S \text{ or } S \subset Z\}.$$

LEMMA 5. *The set $C(S)$ is a chain.*

Proof. Let $S_1, S_2 \in C(S)$. We consider three cases:

(1) $S_1 \subset S$ and $S_2 \subset S$.

Since $\mathscr{S}$ is a T_1-subbase and $X \notin \mathscr{S}$, there exists a $Z \in \mathscr{S}$ such that $S \cap Z = \emptyset$. It follows that $S_1 \cap Z = \emptyset = S_2 \cap Z$. Hence, by (I), $S_1 \subset S_2$ or $S_2 \subset S_1$.

(2) $S_1 \subset S \subset S_2$ or $S_2 \subset S \subset S_1$.

(3) $S \subset S_1$ and $S \subset S_2$.

Since $\mathscr{S}$ is a T_1-subbase and $X \notin \mathscr{S}$, there exist sets Z_1, Z_2 in $\mathscr{S}$ such that $S_1 \cap Z_1 = \emptyset$ and $S_2 \cap Z_2 = \emptyset$. Since $S \subset S_1$ and $S \subset S_2$, we obtain $S \cap Z_1 = \emptyset = S \cap Z_2$ and, by (I), we have $Z_1 \subset Z_2$ or $Z_2 \subset Z_1$. Suppose that $Z_1 \subset Z_2$. It follows that $S_1 \cap Z_1 = \emptyset = S_2 \cap Z_1$ and, by (I), we have $S_1 \subset S_2$ or $S_2 \subset S_1$.

Let us take an equivalence class E of the relation $\sim$.

LEMMA 6. *If S, $T \in E$ and $S \cap T = \emptyset$, then*

$$E = C(S) \cup C(T) \quad \textit{and} \quad C(S) \cap C(T) = \emptyset.$$

Proof. If $\mathscr{S}$ fulfils (I), then $\mathscr{S}$ fulfils (vI). Hence, by Lemma 4, it follows that E fulfils (*). Let us take an element $Z \in E$. If $Z \subset S$ or $S \subset Z$, or $Z \subset T$, or $T \subset Z$, then $Z \in C(S) \cup C(T)$.

Let $Z \cap S = \emptyset$. Since $S \cap T = \emptyset$, we have $Z \subset T$ or $T \subset Z$ by (I).

Let $Z \cup S = X$. Since $\mathscr{S}$ is a T_1-subbase, there exists a $G \in \mathscr{S}$ such that $Z \cap G = \emptyset$. Therefore $G \subset S$. Hence $G \cap T = \emptyset = G \cap Z$ and $T \subset Z$ or $Z \subset T$ by (I).

Suppose, on the contrary, that $C(S) \cap C(T) \neq \emptyset$ and let $Z \in C(S) \cap \cap C(T)$. Then $S \subset Z$ and $T \subset Z$. Since $\mathscr{S}$ is a T_1-subbase, there exists a $G \in \mathscr{S}$ such that $G \cap Z = \emptyset$. Hence $G \cap S = \emptyset = G \cap T$. Consequently, by (I), $S \subset T$ or $T \subset S$, a contradiction.

COROLLARY 1. *The set $\{E(S): S \in C(T)\}$ is a chain in X_E.*

COROLLARY 2. *The set $C(S)$ is an mls in E.*

In the space X_E we define a linear order in the following way:

Let us choose S, T in $\mathscr{S}$ such that $S \cap T = \emptyset$ and $C(S), C(T) \in X_E$. Then we put $C(S)$ as the first element and $C(T)$ as the last element. Let us take two different mls's M and N from E which belong to X_E. There are $Z \in M$ and $F \in N$ with $Z \cap F = \emptyset$. Then one of them belongs to $C(S)$ and the other to $C(T)$. Assume that $Z \in C(S)$ and $F \in C(T)$. Then we let $M < N$.

To see that it is a linear order on X_E it remains to verify that if $M < N$ and $N < H$, then $M < H$, and that if $M < N$, then $N < M$ does not hold.

Let M, N, and H be in X_E and assume that $M < N$ and $N < H$. If $M < N$, then there are sets $Z \in M$ and $F \in N$ with $Z \in C(S)$, $F \in C(T)$, and $Z \cap F = \emptyset$. If $N < H$, then there are sets $Z' \in N$ and $F' \in H$ with $Z' \in C(S)$, $F' \in C(T)$, and $Z' \cap F' = \emptyset$. Since Z and Z' are in $C(S)$, we have $Z \subset Z'$ or $Z' \subset Z$. Since Z' and F are in N, $Z' \cap F \neq \emptyset$. But since $Z \cap F = \emptyset$, $Z \subset Z'$. Hence $Z \cap F' = \emptyset$ and, consequently, we have $M < H$.

Suppose that there are M and N in X_E with $M < N$ and $N < M$. If $M < N$, then there are sets $Z \in M$ and $F \in N$ with $Z \in C(S)$, $F \in C(T)$, and $Z \cap T = \emptyset$. If $N < M$, then there are sets $Z' \in N$ and $F' \in M$ with $Z' \in C(S)$, $F' \in C(T)$, and $Z' \cap F' = \emptyset$. Since Z and Z' are in $C(S)$, we have $Z' \subset Z$ or $Z \subset Z'$. If $Z \subset Z'$, then $Z' \cap F' = \emptyset$. But $Z, F' \in M$, a contradiction. If $Z' \subset Z$, then $Z' \cap F = \emptyset$. But $Z', F \in N$, a contradiction.

THEOREM 3. *Topology on X_E is induced by the linear order $<$.*

Proof. By Lemma 6 we have $E = C(S) \cup C(T)$. Let us consider the set $\{M \in X_E\colon M \leqslant N\}$. If $N \neq C(T)$, then $N \cap C(S) \neq \emptyset$. (In the case $N = C(T)$, $\{M \in X_E\colon M \leqslant N\} = X_E$.) Let us take for each $Z \in N \cap C(S)$ the set $E(Z)$. Then

$$\{M \in X_E\colon M \leqslant N\} = \bigcap \{E(Z)\colon Z \in N \cap C(S)\}.$$

To see this observe that if $M < N$, then there exist sets $F \in M$ and $F' \in N$ such that $F \in C(S)$ and $F' \in C(T)$, and $F \cap F' = \emptyset$. Since $Z \in C(S)$ and $Z \in N$, we have $F \subset Z$. Therefore $Z \in M$, and hence $M \in E(Z)$ for each $Z \in N \cap C(S)$. Thus

$$\{M \in X_E\colon M \leqslant N\} \subset \bigcap \{E(Z)\colon Z \in N \cap C(S)\}.$$

For the converse suppose, on the contrary, that there exists an $H \in \bigcap \{E(Z)\colon Z \in N \cap C(S)\}$ with $H \notin \{M \in X_E\colon M \leqslant N\}$. Then $N < H$ and, therefore, there are sets $G \in N$ and $G' \in H$ such that $G \in C(S)$, $G' \in C(T)$, and $G \cap G' = \emptyset$. Hence $G \in N \cap C(S)$. But then $G \in H$, a contradiction.

We do the same when we try with the set $\{M \in X_E\colon N \leqslant M\}$.

Let us take $Z \in E$ and the set $E(Z)$. Suppose that $Z \in C(S)$. For each point $x \in X \setminus Z$ there exists a $G_x \in C(T)$ such that $Z \cap G_x = \emptyset$ and $x \in G_x$. The family of those G_x's is enclosed in $C(T)$. These sets form a chain. If we take two different elements from this chain, we can choose a point p lying in their difference. Since $\mathcal{S}$ is a T_1-subbase, for the chosen point we can find in E a set containing this point and disjoint with the smaller (in the sense of our order) set in the given pairs. Thus Z is contained in this element. Let us take the ls $L_p = \{P \in E\colon p \in P\}$ and the set $\{M \in X_E\colon$

$M \leqslant L_p\}$. If $M \in E(Z)$, then $Z \in M$. Let us take for p a set G_x such that $p \in G_x$. Then $G_x \cap Z = \emptyset$, and so $M < L_p$ for each point $p \in X \setminus Z$. Hence

$$E(Z) \subset \bigcap \big\{\{M \in X_E\colon M \leqslant L_p\}\colon p \in X \setminus Z\big\}.$$

Suppose that there exists a $T \in \bigcap \big\{\{M \in X_E\colon M \leqslant L_p\}\colon p \in X \setminus Z\big\}$ with $T \notin E(Z)$. There exists a $P \in T$ such that $Z \cap P = \emptyset$. We show that $P = X \setminus Z$. Since $P \in C(T)$, P is comparable with each G_x. If $P \subsetneqq G_x$, then for each $p \in G_x \setminus P$ there exists an $F \in \mathscr{S}$ such that $P \cap F = \emptyset$ and $p \in F$. Then $L_p < T$, which is a contradiction. Hence $G_x \subset P$ for each $x \in X \setminus Z$, which shows that $P = X \setminus Z$.

Let us take two different points N and N' from $E(Z)$. Since X_E is a T_1-space and $\mathscr{S}_E$ is a T_1-subbase, there exist their disjoint neighborhoods $E(F_N), E(F_{N'}) \in \mathscr{S}_E$, $F_{N'} \cap F_N = \emptyset$. Since E is the sum of chains $C(Z)$ and $C(X \setminus Z)$, F_N belongs to one of them and $F_{N'}$ to the other; say $F_N \in C(Z)$. Then $F_N \subset Z$. Let $F_{N'} \in C(X \setminus Z)$. Then $X \setminus Z \subset F_{N'}$. For each point $y \in Z \setminus F_N$ let us take the set F_y with $y \in F_y$ and $F_y \cap F_N = \emptyset$. Thus $X \setminus Z \subset F_y$. Let us take the linked system

$$R = \{Z\} \cup \{F_y\colon y \in Z \setminus F_N\}.$$

This family is contained in exactly one mls T' (because if T_1 and T_2 are different mls's containing R, then there exist $P_1 \in T_1$ and $P_2 \in T_2$ such that $P_1 \cap P_2 = \emptyset$). Hence $P_1 \in C(Z)$ and $P_2 \in C(X \setminus Z)$ or conversely. Then $Z \subset P_1$ and $X \setminus Z \subset P_2$. But then $P_1 \cap P_2 \neq \emptyset$, since $P_2 \cap Z \neq \emptyset$; a contradiction.

Now we show that for each $N \in E(Z)$ such that $N \neq T$ the relation $N < T'$ holds. Indeed, assuming on the contrary that there is an $N \in E(Z)$ such that $T' < N$ we get sets $H \in T'$ and $H' \in N$ such that $H \in C(S)$, $H' \in C(T)$, and $H \cap H' = \emptyset$. By $H \in C(S) \cap T'$ we have $Z \subset T$. But then $Z \in N$, a contradiction. This shows that $E(Z) = \{M \in X_E\colon M \leqslant T'\}$.

Let us assume that a subbase $\mathscr{S}$ fulfils the condition

(M) For each point $x \in X$ and for each $T \in \mathscr{S}$ such that $x \in T$ there exists an $S \in \mathscr{S}$ such that $x \in S \subset T$, and if $Z \in \mathscr{S}$ and $x \in Z \subset S$, then $Z = S$.

The element S is called *minimal* for x and T.

LEMMA 7. *If $\mathscr{S}$ fulfils* (M), *then so does $\mathscr{S}_E$.*

Proof. The family $\mathscr{S}_E$ is a subbase in X_E. Let $\xi \in X_E$ and take an arbitrary set $E(S) \in \mathscr{S}_E$ such that $\xi \in E(S)$. Then $\bigcap \xi \subset S$. Let $x \in \bigcap \xi$. By (M), there exists a $T \in \mathscr{S}$ such that $T \subset S$ and T is minimal. We will show that $E(T)$ is minimal for ξ and $E(S)$. Since $T \in \mathscr{S}$, we have $\xi \in E(T)$. Suppose that there exists a $Z \in \mathscr{S}$ such that $\xi \in E(Z) \subset E(T)$. It follows that $Z \subset T$. Since $Z \in \xi$, $x \in Z$ and by (M) we have $Z = T$. This completes the proof.

2. In this part of the paper we will give the characterization of 0-dimensional supercompact spaces which fulfil condition (vI).

Let X be a 0-dimensional supercompact space and let $\mathscr{S}$ be a binary subbase for the topology on X which fulfils conditions (vI), (M), and such that if $S \in \mathscr{S}$, then $X \setminus S \in \mathscr{S}$.

Denote by $\mathscr{S}^{\sim}$ the set of all classes of the equivalence relation $\sim$. Any element $E \in \mathscr{S}^{\sim}$ has a corresponding space X_E. By Theorem 1 the space X is homeomorphic to the product of supercompact spaces X_E. By Lemmas 4 and 7, the induced subbases $\mathscr{S}_E$ on X_E fulfil conditions $(*)$ and (M). Let $\mathscr{R}$ be the subset of $\mathscr{S}^{\sim}$ consisting of elements which are finite. The product of elements from $\mathscr{R}$ is homeomorphic to the Cantor discontinuum or is a finite discrete space.

Consider now spaces X_E which have an infinite number of points. An example of such spaces is the one-point compactification of an infinite discrete space.

THEOREM 4. *Assume that a supercompact space X,* $\operatorname{card} X \geqslant \aleph_0$, *has a binary subbase $\mathscr{S}$ which fulfils conditions* $(*)$, (M), *and*

(C) *If $F \in \mathscr{S}$, then $X \setminus F \in \mathscr{S}$.*

Then there exist a finite chain $\{X_k\}$, $k = 0, \ldots, n$, of subsets of the space X and a finite chain of decompositions $\{\mathscr{W}_k\}$ of the sets $X \setminus X_k$ with the following properties:

(1) *$X_i \subset X_k$ for $k \geqslant i$; X_0 is a one-point set.*

(2) *X_k is a nowhere dense subset of X_{k+1}.*

(3) *$\mathscr{W}_i$ is a refinement of $\mathscr{W}_k$ for $i \geqslant k$.*

(4) *If $W \in \mathscr{W}_n$, then W is an isolated point in X (hence X is a compactification of the discrete space $D = \bigcup \{W\colon W \in \mathscr{W}_n\}$).*

For the proof we need some lemmas.

LEMMA 8. *If a space X fulfils the assumptions of Theorem 4 and has infinitely many nonisolated points, then there exist a point $* \in X$ and a decomposition $\mathscr{W}$ of the set $X \setminus \{*\}$ such that infinitely many elements of the decomposition have an accumulation point.*

Proof. Since the set of all accumulation points is infinite, we can take a nonclosed subset K of the set of accumulation points of X. Let $*$ belong to the set $\operatorname{cl} K \setminus K$. For each point $x \neq *$ there exists a $V_x \in \mathscr{S}$ such that $x \in V_x$ and $* \notin V_x$. By assumption (M) we can take it for granted that V_x is the set which fulfils (M) for a pair x, V_x. Let us put

$$\mathscr{R} = \{V_x\colon x \in X \setminus \{*\}\}$$

and choose an arbitrary maximal chain $\mathscr{L}$ in $\mathscr{R}$. For each $V \in \mathscr{L}$, $X \setminus V \in \mathscr{S}$ and $* \in X \setminus V$. Thus by (M) there exists a minimal element $V_{\mathscr{L}} \subset X \setminus V$ for $*$. By $(*)$, $V_{\mathscr{L}}$ is common for all sets $X \setminus V$ whenever $V \in \mathscr{L}$. Hence $\bigcup \mathscr{L} \subset X \setminus V_{\mathscr{L}}$. Let us take an arbitrary pair of chains $\mathscr{L}$ and $\mathscr{L}'$. Then

$X \setminus V_{\mathscr{L}}$ and $X \setminus V_{\mathscr{L}'}$ are, by condition $(*)$, disjoint or equal. We have got the decomposition of the set $X \setminus \{*\}$ into closed-open sets from the subbase $\mathscr{S}$. If, on the contrary, only finitely many elements of the decomposition have accumulation points, then the union of these elements is a closed-open set containing K, and so also its closure. But this is a contradiction with the assumption $* \in \operatorname{cl} K \setminus K$.

LEMMA 9. *Let X fulfil the assumptions of Theorem 4. If $V \in \mathscr{S}$, then V is minimal for some point.*

Proof. Suppose, on the contrary, that there exists a $V \in \mathscr{S}$ which is not minimal for any point. For each $x \in V$ we take a minimal element $V_x \subset V$. Since V is closed-open, there exists a finite family $\{V_1, \ldots, V_n\} \subset \{V_x: x \in V\}$ such that $V = V_1 \cup \ldots \cup V_n$. Hence

$$X \setminus V = \bigcap (X \setminus V_i). \tag{1}$$

We may assume that no element of this family is included in another one. Since $\mathscr{S}$ fulfils condition $(*)$, elements $V_1, \ldots, V_n$ are disjoint. But since each V_i is different from V, the family $R = \{V, X \setminus V_1, \ldots, X \setminus V_n\}$ is linked. Since $\mathscr{S}$ is binary, $\bigcap R \neq \emptyset$. A contradiction with (1).

LEMMA 10. *Let X fulfil the assumptions of Theorem 4. If V is a minimal element for x, and $y \in V$ $(x \neq y)$, then V is not minimal for y.*

Proof. Since $\mathscr{S}$ is a binary subbase consisting of closed-open sets, there exist sets U and H from $\mathscr{S}$ such that $x \in U$, $y \in H$, and $U \cap H = \emptyset$. We can assume that both sets U and H are minimal. Hence we have a pair U, V of different minimal elements for a point x. Then, by $(*)$, $U \cup V = X$. Hence the set $X \setminus U$ is included in V and is different from V which is a neighborhood of a point y.

LEMMA 11. *The subbase $\mathscr{S}$ from Theorem 4 can be modified in such a way that the isolated points of the space X belong to $\mathscr{S}$.*

Proof. Let x be an isolated point which does not belong to $\mathscr{S}$. Let us take an arbitrary minimal set V for x. By Lemma 9, the set $X \setminus V$ is minimal for some point y. Instead of the pair V, $X \setminus V$ take the pair $\{x\}$, $X \setminus \{x\}$. We show that $X \setminus \{x\}$ is minimal for the point y. If it is not true, then there exists a set $W \in \mathscr{S}$, $W \neq X \setminus V$, minimal for y and $X \setminus \{x\}$. Since $\mathscr{S}$ fulfils $(*)$, we have $W \subset X \setminus V$ or $X \setminus V \subset W$. By (M), $X \setminus V \subset W$. Suppose that $W \neq X \setminus V$. Then by Lemma 9 there exists a point p such that W is minimal for p. Hence $p \in W \setminus (X \setminus V)$. Since $p \in V$, by Lemma 10 there exists a $V_p \subset V$, $V_p \neq V$, such that $p \in V_p$ and V_p is minimal for p. The sets W and V_p being minimal for p, by $(*)$ we have $V_p \cup W = X$. Since $x \notin W$, $x \in V_p \subset V$; a contradiction with the assumption that V is minimal for x.

Proof of Theorem 4. Let us consider two cases.

I. The space X has finitely many accumulation points. Then X is the sum of one-point compactifications of discrete spaces.

II. The space X has infinitely many accumulation points. By Lemma 8 there exist such a point $*$ (it will be our space X_0) and a decomposition $\mathscr{W}_0$ of the space $X\setminus X_0$ such that elements of $\mathscr{W}_0$ belong to $\mathscr{S}$. Denote by $\mathscr{W}_i'$ a subfamily of $\mathscr{W}_i$ consisting of infinite elements. Suppose that the pairs X_i, $\mathscr{W}_i$ are defined for $i \leqslant k$. Let us take a subfamily $\mathscr{W}_k' \subset \mathscr{W}_k$ consisting of infinite elements. For each $W \in \mathscr{W}_k'$ we take a point for which W is minimal. Let

$$X_{k+1} = X_k \cup \{p\colon\, p \in W \in \mathscr{W}_k' \text{ and } W \text{ is minimal for } p\}.$$

For each point $y \in W$, $y \neq p$, let us take, by Lemma 10, a minimal set V_y for y and W. Denote by R the family of all these minimal sets. For each maximal chain $\mathscr{L} \subset R$ let us take a family of neighborhoods of a point p of the form $X\setminus V$, where $V \in \mathscr{L}$. Since $\mathscr{S}$ fulfils condition (M), for each $X\setminus V$ there exists a minimal set for p and $X\setminus V$. Since $\mathscr{S}$ fulfils condition $(*)$, this minimal set is one for all $X\setminus V$, where $V \in \mathscr{L}$. Denote this minimal set by $V_{\mathscr{L}}$. By $(*)$, $V_{\mathscr{L}} \cup W = X$. Hence $X\setminus V_{\mathscr{L}} \subset W$. For each pair of chains $\mathscr{L}$ and $\mathscr{L}'$ the sets $X\setminus V_{\mathscr{L}}$ and $X\setminus V_{\mathscr{L}'}$ are equal or disjoint. Hence we have a partition of the set $W\setminus\{p\}$ into elements of $\mathscr{S}$. Now, let us take this partition for all $W \in \mathscr{W}_k'$ and add all elements from $\mathscr{W}_k\setminus\mathscr{W}_k'$. Hence we have the partition of the set $X\setminus X_{k+1}$. The pair $X_{k+1}, \mathscr{W}_{k+1}$ fulfils $(*)$ and (C). We will show that this pair fulfils condition (M). Since $X\setminus X_k = \bigcup \mathscr{W}_k$, where each $W \in \mathscr{W}_k$ is open, the set X_k is closed in X. Take the set $X_{k+1}' = X_{k+1}\setminus X_k$. We will show that X_{k+1}' is dense in X_{k+1}. From the induction hypothesis we infer that X_k' is dense in X_k (X_1 is a one-point compactification of a discrete space). It suffices to show that each point from X_k is an accumulation point for X_{k+1}. Let $x \in X_k'$. Then there exists a $W \in \mathscr{W}_{k-1}$ such that W is minimal for x. Let us take those elements from $\mathscr{W}_k$ which form the partition of the set $W\setminus\{x\}$. Let us take one point from each element of the partition. The set D of all these points is discrete in the induced topology and x is an accumulation point for this set. Hence X_k is nowhere dense in X_{k+1}.

Now we show that after finitely many steps $\bigcup X_k = X$.

Suppose the contrary. We will show that $Y = \bigcup X_n$ is closed in X. Let us take a nonisolated point $y \in X\setminus Y$. For each n there exists a $V_n \in \mathscr{W}_n$ such that $y \in V_n$. Since $\mathscr{W}_{n+1} \prec \mathscr{W}_n$, the family $\{V_n\}$ forms a chain. From condition (M) it follows that there exists a $V \in \mathscr{S}$ such that $y \in V \subset V_n$ and V is minimal for each V_n. From Lemma 10 it follows that V is disjoint with the set consisting of all those points for which the sets V_n are minimal. Hence $V \cap Y = \emptyset$. Since X_n is nowhere dense in X_{n+1}, X_n is nowhere dense in Y. Hence Y is a countable union of nowhere dense subsets, which is a contradiction with the Baire category theorem.

3. This section is a review of theorems in which the subbase concept is used to characterize several spaces.

THEOREM 5 (van Dalen [4]). *A T_1-space X is a product of spaces with the topology induced by linear order if and only if there exists a q-binary T_1-subbase for the topology on X which fulfils conditions* (I) *and* (F).

THEOREM 6 (de Groot and Schnare [9], Szymański and Turzański [14]). *A Hausdorff continuum is a product of linearly ordered Hausdorff continua if and only if there exists a binary subbase for its topology which fulfils condition* (I).

THEOREM 7 (de Groot [8], Szymański and Turzański [14]). *A metrizable continuum X is a Euclidean n-cube* (*Hilbert cube*) *if and only if there exists a binary subbase $\mathscr{S}$ for its topology and* $\operatorname{card}\mathscr{S}/\!\sim\, = n$ $(\operatorname{card}\mathscr{S}/\!\sim\, \geqslant \aleph_0)$.

These three theorems are immediate corollaries to Theorems 1 and 3.

• THEOREM 8 (Szymański and Turzański [14]). *A space X is a Tychonoff cube if and only if it is a dyadic continuum and if there exists a binary subbase for its topology which fulfils condition* (I).

Proof. Dyadic spaces have a topological characterization given by Alexandroff and Ponomarev in [1]. By Theorem 6, X is a product of linearly ordered dyadic continua. By the theorem of Mardešić and Papić [10], each element of the product is a topologically closed segment. Thus X is a Tychonoff cube.

A point $x \in X$ is called a *separation point* of y and z $(y, z \in X)$ if $X \setminus \{x\} = A \cup B$ with $y \in A$, $z \in B$, $A \cap B = \emptyset$, and A, B are open.

A connected space is said to be *tree-like* if every two points of this space have a separation point.

Brouwer and Schrijver [3] and van Mill [11] proved that compact tree-like spaces are supercompact.

THEOREM 9 (van Mill [11]). *Let X be a topological space. Then the following properties are equivalent*:

(1) *X is compact tree-like.*

(2) *X is connected and has a binary normal subbase such that it fulfils condition* (*).

From Theorem 1 and Lemma 4 we obtain

THEOREM 10 (van Mill [11]). *If X is a compact connected space which has a binary subbase fulfilling condition* (vI), *then X is homeomorphic to the product of compact tree-like spaces.*

THEOREM 11. *Assume that a space X has a q-binary T_1-subbase $\mathscr{S}$ which fulfils condition* (F) *and is such that*

(a) *$\mathscr{S}|P$ is a q-binary subbase for each closed subset P of X,*

(b) *if $S \in \mathscr{S}$, then* $\operatorname{cl}(X \setminus S) \in \mathscr{S}$ *and* $\operatorname{card} X \geqslant 4$.

Then the topology on X is generated by a linear order.

Proof. We will show that the subbase $\mathscr{S}$ fulfils condition (I). Let us take three sets S_0, S_1, S_2 such that $S_0 \cap S_1 = \emptyset = S_0 \cap S_2$ and suppose on the contrary that $S_1 \not\subset S_2$ and $S_2 \not\subset S_1$. Let

$$x \in \operatorname{Int}(S_1 \setminus S_2), \quad y \in \operatorname{Int}(S_2 \setminus S_1), \quad z \in \operatorname{Int} S_0.$$

Take the closed set $\{x, y, z\} = Y$. The sets $\{x, z\} = \operatorname{cl}(X \setminus S_2) \cap Y$, $\{y, z\} = \operatorname{cl}(X \setminus S_1) \cap Y$, and $\{x, y\} = \operatorname{cl}(X \setminus S_0) \cap Y$ create an ls, but their intersection is empty, a contradiction.

Now, by Theorems 1 and 3, X is a product of linearly ordered spaces. Suppose that $X = X_1 \times X_2$ and that X_1 and X_2 are linearly ordered spaces. Then one of these spaces, say X_1, has three or more points. Let $a < b < c$, $\{a, b, c\} \subset X_1$, and $x < y$, $\{x, y\} \subset X_2$. Then $\mathscr{S} | \{(a, x), (b, y), (c, x)\}$ is not q-binary.

For $X = X_1 \times X_2 \times \ldots$, let $a_1 < a_2$, $a_1, a_2 \in X_1$, $b_1 < b_2$, $b_1, b_2 \in X_2$, and $c_1 < c_2$, $c_1, c_2 \in X_3$. Then

$$\mathscr{S} | \{(a_1, b_1, c_1, \ldots), (a_1, b_2, c_2, \ldots), (a_2, b_1, c_2, \ldots)\}$$

is not q-binary.

Hence the topology on X is generated by a linear order.

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