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ON CARDINALITIES OF ALGEBRAS OF FORMULAS FOR ω_0-CATEGORICAL THEORIES

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1. Preliminaries. This paper is a by-product of paper [4], so we shall use the same notation. $\boldsymbol{T}$ will denote a theory in a first order language $\boldsymbol{L}$. All discussed theories will be assumed complete. $\mathscr{F}_n(\boldsymbol{T})$ will be the Lindenbaum algebra with respect to $\boldsymbol{T}$ of formulas of $\boldsymbol{L}$ with n free variables. The power of a set A will be denoted by $|A|$. We write $a_n(\boldsymbol{T}) = |\mathscr{F}_n(\boldsymbol{T})|$. The number of atoms in $\mathscr{F}_n(\boldsymbol{T})$ will be denoted by $b_n(\boldsymbol{T})$. If $\boldsymbol{T} = \mathrm{Th}(\mathfrak{A})$, we write $\mathscr{F}_n(\mathfrak{A})$ instead of $\mathscr{F}_n(\boldsymbol{T})$.

Other analogous notations will be used. An element of $\mathscr{F}_n(\boldsymbol{T})$ will be denoted by $\varphi/\boldsymbol{T}$, where φ is a representative of the element under discussion (such an element is an equivalence class of formulas).

By a theorem of Ryll-Nardzewski [2], *$a_n(\boldsymbol{T})$ is finite for every n if and only if $\boldsymbol{T}$ is ω_0-categorical.*

In 1969 Ryll-Nardzewski raised a problem of characterisation of sequences of Boolean algebras $\langle \mathfrak{B}_n : n < \omega\rangle$ such that there exists $\boldsymbol{T}$ such that $\mathfrak{B}_n \simeq \mathscr{F}_n(\boldsymbol{T})$. We will discuss the problem: for which sequences of natural numbers $\langle a_n : n < \omega\rangle$, $a_n = a_n(\boldsymbol{T})$ for some ω_0-categorical $\boldsymbol{T}$? In this case $b_n(\boldsymbol{T}) = \log_2(a_n(\boldsymbol{T}))$, so that we can equivalently look for conditions on sequences $\langle b_n(\boldsymbol{T}) : n < \omega\rangle$.

A relational structure will be called *ω_0-categorical* if its theory is categorical in ω_0.

In the second section we give the characterisation of sequences $\langle b_n(\boldsymbol{T}) : n < \omega\rangle$ for ω_0-categorical Boolean algebras. In the third section we give some simple necessary conditions for the general case. The most interesting result of this section seems to be the following one:

The sequence $\langle b_n(\boldsymbol{T}) : n < \omega\rangle$ is bounded by an exponential function if and only if $\boldsymbol{T}$ is a theory of a finite relational structure.

2. Powers of $\mathscr{F}_n$-algebras for ω_0-categorical Boolean algebras. We recall that a Boolean algebra is *ω_0-categorical* if and only if it has finitely many atoms. We start our discussion with the case of a finite algebra.

We will use the following notation: "card $x = k$" will denote the formula of the language of Boolean algebras which says that "x is a union of k atoms". If $\varepsilon = 1$, then $\varepsilon x = x$; if $\varepsilon = 0$, then $\varepsilon x = -x$, for $\bar{\varepsilon} = \langle \varepsilon_0, \ldots$ $\ldots, \varepsilon_{n-1}\rangle$, $\bar{\varepsilon}\bar{x} = \bigcap \{\varepsilon_i x_i\colon i < n\}$.

The following lemma can be easily proved by the standard inductive procedure (cf. [3]):

LEMMA 1. *For a finite Boolean algebra $\mathfrak{B} \simeq 2^m$, every formula with n free variables is equivalent to some disjunction of formulas of the form*

$$\bigwedge \{\mathrm{card}\, \bar{\varepsilon}\bar{v} = k(\bar{\varepsilon})\colon \bar{\varepsilon} \in 2^n\}. \tag{2.1}$$

As an immediate application of this lemma we obtain

PROPOSITION 1. *$b_n(2^m)$ is equal to the number of all functions k with he domain 2^n and range $m+1$ such that*

$$\sum \{k(\bar{\varepsilon})\colon \bar{\varepsilon} \in 2^n\} = m. \tag{2.2}$$

Proof. By Lemma 1, every atom in $\mathscr{F}_n(2^m)$ can be represented by a formula of form (2.1), and, conversely, every formula of this form which is not false (i. e., for which (2.2) holds) represents some atom in $\mathscr{F}_n(2^m)$.

From Proposition 1 we obtain the following algorithm for computing $b_n(2^m)$. Let $b(n, m)$ be the number of all functions on n, with non-negative integer values, and such that the sum of all values is equal to m. We can extend this definition by putting $b(0, 0) = 1$, and $b(0, m) = 0$ for $m \neq 0$. It is easy to see that

$$b(n+1, m) = \sum_{k=0}^{m} b(n, k). \tag{2.3}$$

By Proposition 1, $b_n(2^m) = b(2^n, m)$, and from formula (2. 3) we can obtain recursive formulas for $b(n, m)$, for fixed m. So, we can obtain the formulas for $b_n(2^m)$.

For example, $b(n, 0) = b(n-1, 0) = \ldots = b(0, 0) = 1$; $b(n, 1)$ $+\ b(n-1, 1)+1$, so $b(n, 1) = n$; $b(n, 2) = b(n-1, 2)+b(n-1, 1)+$ $=b(n-1, 0) = b(n-1, 2)+n$, so $b(n, 2) = n(n+1)/2$.

From these formulas we infer that $b_n(2) = 2^n$, and $b_n(2^2) = (4^n+2^n)/2$. The last formula can be used in estimation of the power of $\mathscr{F}_n$-algebras for finite direct products (see Proposition 2 in [4]).

As in Lemma 1 and Proposition 1, we can determine the number of atoms in $\mathscr{F}_n(\mathfrak{B})$ for atomless Boolean algebra $\mathfrak{B}$. Namely, by a theorem of Skolem [3], from every element of $\mathscr{F}_n(\mathfrak{B})$ one can choose a quantifier-free representative. By an easy induction, one can prove that every formula is equivalent to an alternative of conjunctions of formulas of the form $\bar{\varepsilon}\bar{v} = 0$ or $\bar{\varepsilon}\bar{v} \neq 0$. So, every atom can be represented by a not false formula

$$\bigwedge \{\eta(\bar{\varepsilon})(\bar{\varepsilon}\bar{v} \neq 0)\colon \bar{\varepsilon} \in 2^n\},$$

where, by analogy to the previously used notation, $\eta\varphi = \varphi$ for $\eta = 1$, and $\eta\varphi = \neg\varphi$ for $\eta = 0$. It is evident that such a formula is not false if and only if $\eta(\bar{\varepsilon}) \neq 0$ for some $\bar{\varepsilon} \in 2^n$. So, we have proved the following proposition:

PROPOSITION 2. *For atomless Boolean algebra* $\mathfrak{B}$ $b_n(\mathfrak{B}) = 2^{2^n} - 1$.

As a simple corollary to Propositions 1 and 2 one can prove the following result:

PROPOSITION 3. *If* $\mathfrak{B}$ *is infinite and has a finite number of atoms, say* m, *then the number of atoms in* $\mathscr{F}_n(\mathfrak{B})$ *is equal to* $b(2^n, m)(2^{2^n} - 1)$.

Proof. Every free variable in a formula representing an atom o- $\mathscr{F}_n(\mathfrak{B})$ can be divided into atomic and atomless parts to which the prelf vious propositions can be applied.

3. Some contributions to the problem of C. Ryll-Nardzewski.

PROPOSITION 4. $b_n(T) \geqslant b_k(T) \cdot b_{n-k}(T)$.

Proof. Let $\mathscr{G}$ be a subalgebra of $\mathscr{F}_n(T)$ generated by a set Φ of al classes φ/T for

$$\varphi \leftrightarrow \psi_1(v_0, \ldots, v_{k-1}) \wedge \psi_2(v_k, \ldots, v_{n-1}), \tag{3.1}$$

where ψ_1 is an atom in $\mathscr{F}_k(T)$, and ψ_2 represents some atom in $\mathscr{F}_{n-k}(T)$. Because $\mathscr{G} \subseteq \mathscr{F}_n(T)$, it suffices to show that Φ is a set of atoms of $\mathscr{G}$, and that the power of Φ is equal to $b_k(T) \cdot b_{n-k}(T)$.

First of all, we show that every element of $\mathscr{G}$ is a union of elements of Φ (0 is an empty union). For the proof, it suffices to show that the family of such unions, say $\mathscr{G}'$, is a subalgebra of $\mathscr{G}$ (so it is equal to $\mathscr{G}$). From the definition, $\mathscr{G}'$ is closed under unions, and it is closed under intersections by the distributivity laws. The complement of any element of $\mathscr{G}'$ is an intersection of complements of members of Φ. Since the algebras $\mathscr{F}_k$ and $\mathscr{F}_{n-k}$ are finite, the complement of an element of Φ belongs to $\mathscr{G}'$.

To prove that Φ has the desired power, let us assume that $T \vdash \psi_1 \wedge \psi_2 \to \psi_1' \wedge \psi_2'$. Then $T \vdash \psi_1 \wedge \psi_2 \to \psi_1'$. But T is complete and ψ_1 and ψ_2 have no common variable. Then either $T \vdash \psi_1 \to \psi_1'$ or $T \vdash \neg\psi_2$. But $\psi_1, \psi_1', \psi_2, \psi_2'$ represent atoms in corresponding Boolean algebras. So, $\psi_1/T = \psi_1'/T$ and $\psi_2/T = \psi_2'/T$, which completes the proof.

COROLLARY 1. $b_n(T) \geqslant \big(b_1(T)\big)^n$.

The natural question arises: when in this formula the equality appears. The answer will be given in the next proposition.

PROPOSITION 5. $b_n(\mathfrak{A}) = \big(b_1(\mathfrak{A})\big)^n$ *if and only if* $\mathfrak{A}$ *is finite and every element of its universe is definable.*

Proof. Let every element of a finite $\mathfrak{A}$ be definable, and let φ have n free variables. Let

$$A(\varphi) = \{\langle a_0, \ldots, a_{n-1}\rangle : \mathfrak{A} \models \varphi[a_0, \ldots, a_{n-1}]\}.$$

Then

$$\mathfrak{A} \models \varphi \leftrightarrow \bigvee \{\varphi_0(v_0) \wedge \ldots \wedge \varphi_{n-1}(v_{n-1}) \colon \langle a_0, \ldots, a_{n-1}\rangle \in A(\varphi), \text{ and } \varphi_i \text{ defines } a_i\}.$$

Conversely, let not every element of $\mathfrak{A}$ be definable. Then there exists an atom φ_0 in $\mathscr{F}_1(\mathfrak{A})$ which is satisfiable by at least two different elements of $\mathfrak{A}$. So, $\varphi_0(v_0) \wedge \varphi_0(v_1) \wedge v_0 = v_1$ and $\varphi_0(v_0) \wedge \varphi_0(v_1) \wedge v_0 \neq v_1$ represent two disjoint elements of $\mathscr{F}_2(\mathfrak{A})$. Using the argumentation analogous to the proof of necessity, one can show that $b_2(\mathfrak{A}) \geqslant \big(b_1(\mathfrak{A})\big)^2 + 1$.

COROLLARY 2. *For a finite* $\mathfrak{A}$, $b_n(\mathfrak{A}) \leqslant |\mathfrak{A}|^n$.

This corollary gives an upper bound of the powers of $\mathscr{F}_n$-algebras. To obtain a lower bound, we use the theory of equality. Let, for $m \leqslant \omega$, $\mathfrak{m}$ be a structure with the equality as the only relation, and with the universe $\{k \colon k < m\}$. We start from the obvious lemma:

LEMMA 2. (a) $\mathscr{F}_n(\mathfrak{m}) \subseteq \mathscr{F}_n(\mathfrak{A})$ *for* $|\mathfrak{A}| = m$.

(b) $\varphi/\mathfrak{m}$ *is an atom in* $\mathscr{F}_n(\mathfrak{m})$ *if and only if*

$$\mathfrak{m} \models \varphi \leftrightarrow \bigwedge \{v_i = v_j \colon \langle i, j\rangle \in \varrho\} \wedge \bigwedge \{v_i \neq v_j \colon \langle i, j\rangle \notin \varrho\},$$

where ϱ *is an equivalence relation on* n *with at most* m *equivalence classes.*

PROPOSITION 6. (a) $b_n(\mathfrak{m})$ *is equal to the number of all equivalence relations on* n *with at most* m *equivalence classes.*

(b) $$b_n(\mathfrak{m}) = \sum_{k=0}^{n-1} \binom{n-1}{k} b_{n-k-1}(\mathfrak{m}-1),$$

$$b_n(0) = 0 \textit{ for } n \neq 0,\ b_0(\mathfrak{m}) = 1.$$

(c) $$b_n(\omega) = b_n(\mathfrak{n}) = \sum_{k=0}^{n-1} \binom{n-1}{k} b_{n-k-1}(\omega).$$

Proof. (a) is a simple reformulation of Lemma 2 (b).

(c) is a consequence of (b) and of the fact that if $m \geqslant n$, then $b_n(\mathfrak{m}) = b_n(\mathfrak{n})$.

(b) To determine an equivalence relation ϱ on n, it suffices to determine the class of the element $n-1$. Let this class have k elements different from $n-1$. Then there are $\binom{n-1}{k}$ possibilities of such a choice. If this class is determined, then one has to define the partition of the remainder into at most $m-1$ classes.

We collect the results of this section in the following theorem:

THEOREM. (a) *If* $|\mathfrak{A}| = m < \omega$, *then* $b_n(\mathfrak{m}) \leqslant b_n(\mathfrak{A}) \leqslant m^n$.

(b) *If* $|\mathfrak{A}| = \omega$, *then* $b_n(\omega) \leqslant b_n(\mathfrak{A})$, *and there is no integer* d *such that* $b_n(\mathfrak{A}) \leqslant d^n$ *for every* $n < \omega$.

Proof. (a) and the first part of (b) follow from Lemma 2(a) and Corollary 2. The second part of (b) follows from the inequality $b_n(\omega) \geqslant (n-1) b_{n-2}(\omega)$. Because $b_0(\omega) = 1$, then $b_n(\omega) \geqslant (n-1)!$.

Not only examples of countably categorical Boolean algebras and of theory of equality, but also a few other examples of countably categorical theories seem to give evidence for the validity of the following conjecture:

CONJECTURE. For a decidable countably categorical theory T, $a_n(T)$ s a recursive function. (**P 818**)

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