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COUNTABLE LOCALLY CONNECTED URYSOHN SPACES

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The first example of a countable (i.e., countably infinite) connected Hausdorff space was given by Urysohn in [11], and many others have been constructed since — for instance in [1], [2] and [3]. Some have additional topological properties: the one in [7] has a dispersion point, the one in [6] is homogeneous, and those in [5] and [8] are locally connected. A somewhat simpler example of this kind is the following (essentially a modification of [1]): Let U_1, $U_2, \ldots$ be an enumeration of the rational open intervals of the real line. Choose a countable collection P of disjoint pairs of (distinct) real numbers such that, given positive integers i and j, there is at least one pair in P having one of its elements in U_i and the other in U_j. Let O_{ij} denote the collection of all pairs p in P such that $p \subset U_i \cap U_j$. Then $\{O_{ij} | i, j = 1, 2, \ldots\}$ is a basis for a topology on P, making P a locally connected, connected countable Hausdorff space.

All the spaces so far mentioned are just barely Hausdorff, in the sense that in each there are distinct points a, b such that, if U and V are neighborhoods of a and b respectively, then $\overline{U} \cap \overline{V} \neq \emptyset$. In other words, they do not satisfy the Urysohn separation axiom. But it is possible to give examples which are Urysohn spaces too. Hewitt [4] gave the first example of a countable connected Urysohn space; one with a dispersion point is in [9], and [10] provides a homogeneous one. It is the purpose of this paper to construct a locally connected one. In fact, as we shall see later, the Urysohn separation axiom can be replaced by any of $\aleph_1$ progressively stronger separation axioms, though not, of course, by regularity [11].

First we construct a countable connected Urysohn space S, which is a slight modification of the space due to Roy in [9].

The points of S. Let C_0, C_1, $C_{-1}, \ldots$ be pairwise disjoint sets of rational numbers, each of which is dense in the reals. The set S will consist of two ideal points, $a = (0, +\infty)$ and $b = (0, -\infty)$, together with the set E_0 of all points of the Cartesian plane of the form (x, y), where $y \in Z$

(the set of all integers) and $x \in C_y$. The points a and b will be called the "endpoints" of S, and also of E_0.

Neighborhoods of points in S. For $p = (x, y) \in E_0$, and $\varepsilon > 0$, let

$$D_\varepsilon(p) = \{(z, w) \mid w = y; z \in C_y; |z-x| < \varepsilon\}$$

when y is even, and

$$D_\varepsilon(p) = \{(z, w) \mid w = y, y+1 \text{ or } y-1; z \in C_w, |z-x| < \varepsilon\}$$

when y is odd. Further, for each $\varepsilon > 0$, let

$$D_\varepsilon(a) = \{a\} \cup \{(z, w) \mid w \geqslant 2[1/\varepsilon]; w \in Z; z \in C_w\}$$

and

$$D_\varepsilon(b) = \{b\} \cup \{(z, w) \mid w \leqslant -2[1/\varepsilon]; w \in Z; z \in C_w\}.$$

These neighborhoods of points of S form a basis for the topology of S. As in [9], S is connected; in fact, it is easy to see directly that S is the only open and closed subset of S which contains a. (In the same way one sees that each $D_\varepsilon(a)$ is connected, and similarly so is each $D_\varepsilon(b)$.) If p and q are distinct points of $E_0 = S - \{a, b\}$, and $\varepsilon > 0$ is less than half the difference between their abscissas, then $\bar{D}_\varepsilon(p) \cap \bar{D}_\varepsilon(q) = \emptyset$. Furthermore, S is regular at both a and b. Hence S is a Urysohn space.

We next obtain a countable, connected, locally connected Urysohn space X.

The points of X. As before, we write E_0 for $S - \{a, b\}$, where S is the space just constructed. Put $G_0 = \{E_0\}$, and choose recursively a sequence $\{G_i \mid i = 1, 2, \ldots\}$ such that each G_i is a countable collection of pairwise disjoint copies $E_i(p, q)$ of E_0, one for each pair of (distinct) points p, q of each E of G_{i-1}. We further require that all the sets G_i^* $(i = 0, 1, 2, \ldots)$ be pairwise disjoint, where G_i^* denotes $\bigcup G_i$. We write $S_i(p, q) = E_i(p, q) \cup \{p, q\}$, and regard $S_i(p, q)$ as a copy of S, in which the points p, q correspond to the endpoints a, b of S. More precisely, we choose a 1-1 map θ_i^{pq} of S onto $S_i(p, q)$ such that $\theta_i^{pq}(a) = p$ and $\theta_i^{pq}(b) = q$. (For $i = 0$ we use instead the identity map θ_0 of E_0.) To simplify the notation, however, we shall omit explicit reference to the copying maps θ_i^{pq}, speaking (for instance) of "$D_\varepsilon(x)$ of $E_i(p, q)$" as short for $\theta_i^{pq}\big(D_\varepsilon\big((\theta_i^{pq})^{-1}(x)\big)\big) \cap E_i(p, q)$.

Now let $X = \bigcup\{G_i^* \mid i = 0, 1, 2, \ldots\}$. Clearly, X is countable.

Neighborhoods of points of X. Let x denote a point of X. There exists a unique non-negative integer n such that x belongs to some element E (necessarily unique) of G_n; here E is either E_0 or some $E_n(r, s)$, where $r, s \in E'$ for some $E' \in G_{n-1}$. For each $\varepsilon > 0$, define $N_\varepsilon(x)$ to be the smallest subset of X which satisfies the following three conditions:

(1) $N_\varepsilon(x) \supset D_\varepsilon(x)$ of E.

(2) If $i > n$, and p, q are distinct points of some element of G_{i-1}, and if $p \in N_\varepsilon(x)$, then $N_\varepsilon(x) \supset D_\varepsilon(p)$ of $S_i(p, q)$.

(3) If i, p, q are as in (2), and if both p and $q \in N_\varepsilon(x)$, then $N_\varepsilon(x) \supset S_i(p, q)$.

There is an obvious inductive construction for $N_\varepsilon(x)$, the first two steps of which are the following:

(1′) x is a non-endpoint of some copy of $S - \{a, b\}$; so put the points of $D_\varepsilon(x)$ of S into $N_\varepsilon(x)$;

(2′) if $x \in G_n^*$, and a copy C of $S - \{a, b\}$ belonging to G_{n+1} has an end-point in $D_\varepsilon(x)$, then the points (of the copy in C) of the ε-neighborhood of this endpoint are put into $N_\varepsilon(x)$, with the understanding that if *both* endpoints of C are in $N_\varepsilon(x)$, then *all* the points of C are put into $N_\varepsilon(x)$. Do this for all such copies C in G_{n+1}, and proceed to G_{n+2}.

This process is very much like Urysohn's in [11], except for the condition of putting all of C into the neighborhood when both its end-points belong to the neighborhood. It is precisely this addition that makes X locally connected.

The sets $N_\varepsilon(x)$ form a basis for a topology on X. It is easy to see that each subspace $S_i(p, q)$ becomes homeomorphic to S (more precisely, each map θ_i^{pq} is a homeomorphism).

X **is connected and locally connected.** Each $N_\varepsilon(x)$ is connected since, after steps (1′) and (2′) above, we have a connected set to which the later steps will merely attach further connected sets as we pass successively to $G_{n+2}, G_{n+3}, \ldots$ Also, while G_0^* is totally disconnected because its end-points have been removed (cf. [9]), $G_0^* \cup G_1^*$ *is* connected; and it follows easily that X is connected.

X **is a Urysohn space.** Suppose that x, y are distinct points of X. If for some i they both belong to G_i^* (whether to the same copy of E_0 in G_i or not), it is clear that we can make $N_\varepsilon(x)$ and $N_\varepsilon(y)$ have disjoint closures by taking ε small enough if x and y belong to the same copy $E_i(p, q)$; ε can be arbitrary if they do not. So we may assume $x \in E_i \in G_i$ and $y \in E_j \in G_j$, where $i < j$. There is a unique sequence $E_j, E_{j-1}, \ldots, E_{i+1}$ such that, for each n between $i+1$ and j inclusive, we have (i) $E_n \in G_n$, (ii) if $n > i+1$, both endpoints of E_n are in E_{n-1}. First suppose $j \geqslant i+2$, let the endpoints of E_{i+2} be p and q (they lie in E_{i+1}), and let the end-points of E_{j+1} be r and s. There exists $\varepsilon > 0$ such that the four sets $D_\varepsilon(r)$, $D_\varepsilon(s)$, $D_\varepsilon(p)$, $D_\varepsilon(q)$ of S_{i+1} have pairwise disjoint closures in S_{i+1} (where S_{i+1} is $E_{i+1} \cup \{r, s\}$). If also ε is small enough so that $\overline{D}_\varepsilon(x)$ in E_i does not contain both r and s, then $\overline{N}_\varepsilon(x) \cap \overline{N}_\varepsilon(y) = \emptyset$. Finally, if $j = i+1$, essentially the same argument applies if we take $p = q = y$, with r and s as before. So X is a Urysohn space.

Remarks. The space X is clearly first (and hence second) countable. We do not know whether it is homogeneous or not (**P 707**). (The points of E_0 with even ordinates appear to be different from those with odd ordinates.) But in any case, Shimrat's technique [10] can be applied to X so as to produce a homogeneous space while preserving the other stated properties of X.

Stronger separation axioms. It is well known [11] that a connected countable space cannot be regular. We have attempted to see how strong a separation axiom it can be made to satisfy, by formulating a transfinite hierarchy of separation axioms as follows. Let α be an ordinal number, and consider the following property of a space:

(P_α) If x and y are distinct points, there exist open sets U_β ($\beta \leqslant \alpha$) such that (1) $x \in U_0$, (2) if $\delta < \gamma \leqslant \alpha$, then $\overline{U}_\delta \subset U_\gamma$, and (3) $y \notin U_\alpha$.

If $\alpha > \beta$, clearly P_α implies P_β. For positive integers n, P_n is equivalent to Viglino's $\overline{T}_n$ [12]; in particular, P_2 is equivalent to the Urysohn separation axiom, and P_1 to the Hausdorff axiom (T_2); while P_0 is equivalent to T_1. P_Ω seems to be too strong to be of much interest (and clearly cannot be satisfied by any countable connected space). However, all completely regular spaces satisfy P_α for all countable ordinals α, and all regular spaces satisfy P_α for all finite α. So does our space S, as is not hard to see; and this carries over to our space X. That is, X has property P_n for each $n = 1, 2, \ldots$

We conclude by sketching a proof of the fact that, for each countable ordinal α, there exists a countable, connected, locally connected, second countable space X which satisfies P_α but not $P_{\alpha+1}$. We suppose that α is a limit ordinal; the other case is treated quite similarly.

The first step is the construction of an analog S_α of the space S. Let Y be the set of all symbols $\pm\beta$, where β is an ordinal less than α; we give Y the obvious linear ordering, and distinguish its elements into "even" and "odd" in the natural way (that is, $\pm\beta$ is even if β is of the form $\lambda + n$, where λ is either 0 or a limit ordinal, and $n = 0, 2, 4, \ldots$). Choose a countable family $\{C_y \mid y \in Y\}$ of pairwise disjoint countable dense sets of real numbers, none of them containing 0. The set S_α will consist of two "endpoints" $a = (0, \alpha)$ and $b = (0, -\alpha)$, together with the set of all ordered pairs (x, y), where $y \in Y$ and $x \in C_y$. Neighborhoods are defined much as for S, except that the endpoints are given smaller neighborhoods. In detail, suppose $p = (x, y) \in S$ and $\varepsilon > 0$. Then, if y is odd,

$$D_\varepsilon(p) = \{(z, w) \mid w = y, y-1 \text{ or } y+1;\ z \in C_w;\ |z-x| < \varepsilon\};$$

if y is even but not $\pm$ a limit ordinal,

$$D_\varepsilon(p) = \{(z, w) \mid w = y;\ z \in C_w;\ |z-x| < \varepsilon\};$$

if y is a limit ordinal (this includes the case $p = a$), then for each even ordinal $\beta < y$

$$D_{\varepsilon\beta}(p) = \{(z, w) \mid \beta \leqslant w \leqslant y;\; z \in C_w;\; |z-x| < \varepsilon\}$$

(when $p = a$ we adjoin the point a to this); finally, when $-y$ is a limit ordinal, $D_{\varepsilon\beta}(p)$ is defined symmetrically (replace $\beta \leqslant w \leqslant y$ by $-\beta \geqslant w \geqslant y$). These sets form a basis for the topology on S_a. To see that P_a is satisfied, we note that a and b can be "separated" in the desired manner by taking $U_a = S_a - \{b\}$ and, for each ordinal $\beta < a$,

$$U_\beta = \{(x, y) \in S_a \mid y \geqslant -(\lambda + 2n)\}, \quad \text{where } \beta = \lambda + n;$$

and b and a can be separated in the symmetric manner. For any other pair of points, we separate their abscissas in the real line, and apply the inverse of the projection map $(x, y) \to x$.

The space S_a is not connected. However, one can show that S_a does not satisfy P_{a+1} with respect to the points a and b; and in particular, S_a is not the union of two disjoint open and closed sets, one containing a and the other containing b. This is enough to ensure that the space X_a, constructed from S_a exactly as X was constructed from S, is connected and locally connected, though the argument is less simple than before. Finally, it is easy to see that X_a inherits the property P_a from S_a, and clearly X_a cannot have the property P_{a+1} since its subspace S_a does not have it.

We have been unable to construct a countable connected, locally connected space which has all the properties P_a for all countable ordinals a simultaneously. It is not necessary to insist on local connectedness here, since the above method will derive a locally connected example from one which is merely connected.

QUESTION. Does there exist a countable connected space $\tilde{X}$ such that, if x and y are distinct points of $\tilde{X}$ and a is a countable ordinal, there exist open sets U_β $(\beta \leqslant a)$ so that (1) $x \in U_0$, (2) if $\delta < \gamma \leqslant a$, then $\overline{U}_\delta \subset U_\gamma$, and (3) $y \notin U_a$? (**P 708**).

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